ENGINEERING PROBABILITY

A formula by Poincaré

On some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ consider any collection of events $A_1, \ldots, A_n$ in $\mathcal{F}$. The formula by Poincaré provides an expression for the probability of their union $\cup_{k=1}^n A_i$. This formula generalizes the basic fact that

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B], \quad A, B \in \mathcal{F} \tag{1.1}$$

obtained by elementary arguments (Fact 1.23 for details). The formula by Poincaré states that

$$\mathbb{P}[\cup_{k=1}^n A_i] = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{1 \le i_1 < \ldots < i_k \le n} \mathbb{P}[A_{i_1} \cap \ldots \cap A_{i_k}] \right). \tag{1.2}$$

We express this formula in the following alternate form

$$\mathbb{P}[\cup_{k=1}^n A_i] = \sum_{\ell=1}^n (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n\}):\ |J|=k} \mathbb{P}[\cap_{j \in J} A_j] \right), \tag{1.3}$$

said form being notationally more convenient to establish it.

A proof

The proof proceeds by induction on $n = 1, \ldots$: The *induction hypothesis* assumes that the formula by Poincaré holds for *any* collection of k events in $\mathcal{F}$ with $k = 2, \ldots, n$.

Basis step: The case $n = 2$ is known to hold – See (1.1) above.

Induction step: Assume that for some $n \ge 2$, (1.3) holds for any collection $\{A_\ell,\ \ell = 1, 2, \ldots, k\}$ of events in in $\mathcal{F}$ with $k = 2, \ldots, n$. We shall show below that (1.3) holds for any collection $\{A_\ell,\ \ell = 1, 2, \ldots, k\}$ of events in $\mathcal{F}$ with $k = 2, \ldots, n, n+1$. Obviously we need only consider the case $k = n + 1$.

The point of departure is the observation that

$$\cup_{i=1}^{n+1} A_i = (\cup_{i=1}^n A_i) \cup A_{n+1}$$

which holds by virtue of the associativity of the union operation. By (1.1) (with $A = \cup_{k=1}^n A_i$ and $B = A_{n+1}$) we get

$$\begin{aligned} \mathbb{P}\left[\cup_{i=1}^{n+1} A_i\right] &= \mathbb{P}[(\cup_{i=1}^n A_i) \cup A_{n+1}] \\ &= \mathbb{P}[\cup_{i=1}^n A_i] + \mathbb{P}[A_{n+1}] - \mathbb{P}[(\cup_{i=1}^n A_i) \cap A_{n+1}] \\ &= \mathbb{P}[\cup_{i=1}^n A_i] + \mathbb{P}[A_{n+1}] - \mathbb{P}[\cup_{i=1}^n (A_i \cap A_{n+1})] \end{aligned} \tag{1.4}$$

since the union and intersection operations are distributive with respect to each other.

Using the induction hypothesis with the collection $A_1, \ldots, A_n$ we get

$$\mathbb{P}\left[\cup_{i=1}^n A_i\right] = \sum_{k=1}^n (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n\}):\ |J|=k} \mathbb{P}\left[\cap_{j\in J} A_j\right] \right). \tag{1.5}$$

On the other hand, the induction hypothesis applied to the collection $A_1 \cap A_{n+1}, \ldots, A_n \cap A_{n+1}$ yields

$$\begin{aligned} &\mathbb{P}\left[\cup_{i=1}^n \left(A_i \cap A_{n+1}\right)\right] \\ &= \sum_{k=1}^n (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n\}):\ |J|=k} \mathbb{P}\left[\cap_{j\in J} \left(A_j \cap A_{n+1}\right)\right] \right) \\ &= \sum_{k=1}^n (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n\}):\ |J|=k} \mathbb{P}\left[\cap_{j\in J\cup\{n+1\}} A_j\right] \right) \\ &= \sum_{k=2}^{n+1} (-1)^{k} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n,n+1\}):\ |J|=k-1,\ n+1\in J} \mathbb{P}\left[\cap_{j\in J} A_j\right] \right) \end{aligned} \tag{1.6}$$

and we conclusion

$$\begin{aligned} &-\mathbb{P}\left[\cup_{i=1}^n \left(A_i \cap A_{n+1}\right)\right] \\ &= \sum_{k=2}^{n+1} (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n,n+1\}):\ |J|=k-1,\ n+1\in J} \mathbb{P}\left[\cap_{j\in J} A_j\right] \right) \end{aligned} \tag{1.7}$$

follows.

Going back to (1.4) we note that

$$\begin{aligned} &\mathbb{P}\left[A_{n+1}\right] + \mathbb{P}\left[\cup_{i=1}^n A_i\right] \\ &= \mathbb{P}\left[A_{n+1}\right] + \sum_{k=1}^n (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n\}):\ |J|=k} \mathbb{P}\left[\cap_{j\in J} A_j\right] \right) \\ &= \sum_{i=1}^{n+1} \mathbb{P}\left[A_i\right] + \sum_{k=2}^n (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n\}):\ |J|=k} \mathbb{P}\left[\cap_{j\in J} A_j\right] \right). \end{aligned} \tag{1.8}$$

Using the expression (1.4) for $\mathbb{P}\left[\cup_{i=1}^{n+1} A_i\right]$ we conclude that this quantity is the sum of three terms, namely

$$\sum_{i=1}^{n+1} \mathbb{P}\left[A_i\right], \tag{1.9}$$

$$\sum_{k=2}^n (-1)^{k+1} \left(\sum_{J \in \mathcal{P}(\{1,\ldots,n\}):\ |J|=k} \mathbb{P}\left[\cap_{j\in J} A_j\right] \right) \tag{1.10}$$

and

$$\sum_{k=2}^{n+1}(-1)^{k+1}\left(\sum_{J\in\mathcal{P}(\{1,\ldots,n,n+1\}):\ |J|=k-1,\ n+1\in J}\mathbb{P}\left[\cap_{j\in J}A_j\right]\right). \tag{1.11}$$

For the first of these terms we note that

$$\sum_{i=1}^{n+1}\mathbb{P}\left[A_i\right]=(-1)^{k+1}\left(\sum_{J\in\mathcal{P}(\{1,\ldots,n+1\}):\ |J|=k}\mathbb{P}\left[\cap_{j\in J}A_j\right]\right)\quad\text{with } k=1 \tag{1.12}$$

while adding the two last terms combine to yield

$$\begin{aligned}&\sum_{k=2}^{n}(-1)^{k+1}\left(\sum_{J\in\mathcal{P}(\{1,\ldots,n\}):\ |J|=k}\mathbb{P}\left[\cap_{j\in J}A_j\right]\right)\\&\qquad+\sum_{k=2}^{n+1}(-1)^{k+1}\left(\sum_{J\in\mathcal{P}(\{1,\ldots,n,n+1\}):\ |J|=k-1,\ n+1\in J}\mathbb{P}\left[\cap_{j\in J}A_j\right]\right)\\&=\sum_{k=2}^{n+1}(-1)^{k+1}\left(\sum_{J\in\mathcal{P}(\{1,\ldots,n+1\}):\ |J|=k}\mathbb{P}\left[\cap_{j\in J}A_j\right]\right).\end{aligned} \tag{1.13}$$

This last step can be argued as follows: Any subset J of $\{1,\ldots,n+1\}$ of size k is either a subset of $\{1,\ldots,n\}$ of size k (and is counted in the first sum) or is a subset of $\{1,\ldots,n+1\}$ which contains $n+1$ (and is counted in the second sum) – In the latter case, such a subset J of size k will be realized as a subset of $\{1,\ldots,n\}$ of size $k-1$ to which $n+1$ is appended.

Collecting terms we conclude that

$$\mathbb{P}\left[\cup_{k=1}^{n+1}A_i\right]=\sum_{k=1}^{n+1}(-1)^{k+1}\left(\sum_{J\in\mathcal{P}(\{1,\ldots,n+1\}):\ |J|=k}\mathbb{P}\left[\cap_{j\in J}A_j\right]\right), \tag{1.14}$$

and the induction step is established.
