
ENGINEERING PROBABILITY

HOMEWORK # 1: Posted on 01/24/2018

Please work out the (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem **1.55** (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). Answers to the problems in BT can be found at <http://www.athenasc.com/probbook.html>.

1. _____
Let S be a set. For arbitrary subsets A and B of S , show the following versions of the De Morgan's Laws:

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c.$$

2. _____
Problem **1.2** (BT)

3. _____
Problem **1.3** (BT)

4. _____
Problem **1.4** (BT)

With A and B being sets, recall that a mapping $g : A \rightarrow B$ is said to be

- *one-to-one* if $g(a) = g(a')$ with a and a' in A , then $a = a'$ necessarily. A one-to-one mapping is also called an *injective* mapping.
- *onto* if for every b in B there exists at least one element a in A such that $b = g(a)$ – In particular, $g(A) = B$ where $g(A) \equiv \{g(a) : a \in A\}$. A mapping that is onto is also called a *surjective* mapping.
- *bijective* (or is a *bijection*) if it is both injective and surjective, or equivalently, one-to-one and onto.

Recall that a set S is *countable* if there exists a one-to-one mapping $f : S \rightarrow \mathbb{N}$ – There may exist many such mappings. Countable sets come in two distinct flavors:

- The set S is *finite* if there exists a positive integer n in \mathbb{N} such that the subset $f(S) \equiv \{f(x) : x \in S\}$ of \mathbb{N} has exactly n (distinct) elements. In that case we say that its *cardinality* is n , and we write $|S| = n$.
- The set S is said to be *countably infinite* if for any one-to-one mapping $f : S \rightarrow \mathbb{N}$, there exists no positive integer n such that $f(S)$ has exactly n (distinct) elements. In that case we say that its cardinality is \aleph_0 (pronounced Aleph zero), and we write $|S| = \aleph_0$.

A set S that is not countable is said to be *uncountable*.

5. _____

First a definition: Two sets A and B are said to be *equipotent* if there exists a bijection $g : A \rightarrow B$.

Show that a set S is finite (resp. countably infinite) if and only if it is equipotent with the set $\{1, \dots, n\}$ for some integers in \mathbb{N} (resp. \mathbb{N}).

6. _____

Let S be a countable set.

6.a If S is a countable set, show that any subset of S is also countable.

6.b Let A be a subset of S . Is it always true that if A is countable, then S must necessarily be countable? Either prove the assertion or give a counter-example.

7. _____

Let S be a countable set that is finite as defined above. Show that if $f : S \rightarrow \mathbb{N}$ and $g : S \rightarrow \mathbb{N}$ are two one-to-one mappings such that $f(S)$ has exactly n_f elements and $g(S)$ has exactly n_g elements, then $n_f = n_g$. In other words, the definition of the cardinality of a finite set is well posed. In particular, it follows that we can always think of the finite set S as being enumerated as $\{a_1, \dots, a_n\}$ where a_1, \dots, a_n are the distinct elements of S .

8. _____

Assume the two sets A and B to be equipotent.

8.a Show that A is finite if and only if B is finite.

8.b Show that A is countably infinite if and only if B is countably infinite.

8.c Show that A is uncountable if and only if B is uncountable.

9. _____

A set S has the following property: There exist one-to-one mappings $g : \mathbb{N} \rightarrow A$ and $h : A \rightarrow \mathbb{N}$. Show that A is countably infinite.

10.

Ninety students, including Joe and Jane, are to be split into three classes of equal size. How often do Joe and Jane end up in the same class?
