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## ENGINEERING PROBABILITY

### HOMEWORK # 1: Posted on 01/24/2018

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Please work out the (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem 1.55 (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). Answers to the problems in BT can be found at <http://www.athenasc.com/probbook.html>.

1. \_\_\_\_\_

Let  $S$  be a set. For arbitrary subsets  $A$  and  $B$  of  $S$ , show the following versions of the De Morgan's Laws:

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c.$$

2. \_\_\_\_\_

Problem 1.2 (BT)

3. \_\_\_\_\_

Problem 1.3 (BT)

4. \_\_\_\_\_

Problem 1.4 (BT)

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With  $A$  and  $B$  being sets, recall that a mapping  $g : A \rightarrow B$  is said to be

- *one-to-one* if  $g(a) = g(a')$  with  $a$  and  $a'$  in  $A$ , then  $a = a'$  necessarily. A one-to-one mapping is also called an *injective* mapping.
- *onto* if for every  $b$  in  $B$  there exists at least one element  $a$  in  $A$  such that  $b = g(a)$   
– In particular,  $g(A) = B$  where  $g(A) \equiv \{g(a) : a \in A\}$ . A mapping that is onto is also called a *surjective* mapping.
- *bijective* (or is a *bijection*) if it is both injective and surjective, or equivalently, one-to-one and onto.

Recall that a set  $S$  is *countable* if there exists a one-to-one mapping  $f : S \rightarrow \mathbb{N}$  – There may exist many such mappings. Countable sets come in two distinct flavors:

- The set  $S$  is *finite* if there exists a positive integer  $n$  in  $\mathbb{N}$  such that the subset  $f(S) \equiv \{f(x) : x \in S\}$  of  $\mathbb{N}$  has exactly  $n$  (distinct) elements. In that case we say that its *cardinality* is  $n$ , and we write  $|S| = n$ .
- The set  $S$  is said to be *countably infinite* if for any one-to-one mapping  $f : S \rightarrow \mathbb{N}$ , there exists no positive integer  $n$  such that  $f(S)$  has exactly  $n$  (distinct) elements. In that case we say that its cardinality is  $\aleph_0$  (pronounced Aleph zero), and we write  $|S| = \aleph_0$ .

A set  $S$  that is not countable is said to be *uncountable*.

**5.** \_\_\_\_\_

First a definition: Two sets  $A$  and  $B$  are said to be *equipotent* if there exists a bijection  $g : A \rightarrow B$ .

Show that a set  $S$  is finite (resp. countably infinite) if and only if it is equipotent with the set  $\{1, \dots, n\}$  for some integers in  $\mathbb{N}$  (resp.  $\mathbb{N}$ ).

**6.** \_\_\_\_\_

Let  $S$  be a countable set.

**6.a** If  $S$  is a countable set, show that any subset of  $S$  is also countable.

**6.b** Let  $A$  be a subset of  $S$ . Is it always true that if  $A$  is countable, then  $S$  must necessarily be countable? Either prove the assertion or give a counter-example.

**7.** \_\_\_\_\_

Let  $S$  be a countable set that is finite as defined above. Show that if  $f : S \rightarrow \mathbb{N}$  and  $g : S \rightarrow \mathbb{N}$  are two one-to-one mappings such that  $f(S)$  has exactly  $n_f$  elements and  $g(S)$  has exactly  $n_g$  elements, then  $n_f = n_g$ . In other words, the definition of the cardinality of a finite set is well posed. In particular, it follows that we can always think of the finite set  $S$  has being enumerated as  $\{a_1, \dots, a_n\}$  where  $a_1, \dots, a_n$  are the distinct elements of  $S$ .

**8.** \_\_\_\_\_

Assume the two sets  $A$  and  $B$  to be equipotent.

**8.a** Show that  $A$  is finite if and only if  $B$  is finite.

**8.b** Show that  $A$  is countably infinite if and only if  $B$  is countably infinite.

**8.c** Show that  $A$  is uncountable if and only if  $B$  is uncountable.

**9.** \_\_\_\_\_

A set  $S$  has the following property: There exist one-to-one mappings  $g : \mathbb{N} \rightarrow A$  and  $h : A \rightarrow \mathbb{N}$ . Show that  $A$  is countably infinite.

**10.** \_\_\_\_\_

Ninety students, including Joe and Jane, are to be split into three classes of equal size. How often do Joe and Jane end up in the same class?

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