ENEE 324-01 / SPRING 2018

HOMEWORK #10

Solutions

① $X : \Omega \to \mathbb{R}$, $X \sim N(0,1)$, $Y : \Omega \to \mathbb{R}$

$$Y = \begin{cases} 0 & \text{if } X \le 0 \\ 1 & \text{if } X > 0 \end{cases}$$

$$\mathbb{P}[Y=0] = \mathbb{P}(X \le 0] = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \quad \text{— ①}$$

$$\mathbb{P}[Y=1] = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \quad \left(\mathbb{P}(X>0) \right] \quad \text{— ②}$$

Note that,

$$\mathbb{P}[Y=0] + \mathbb{P}(Y=1) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = 1 \quad \left(\int_{\mathbb{R}} f_X(x)\,dx = 1 \right)$$

and

$$\mathbb{P}(Y=0] = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \int_{\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}(-dy) \quad (\text{by } y = -x)$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = \mathbb{P}(Y=1) \quad (\text{from } ②)$$

Hence,

$$\mathbb{P}(Y=0) = \mathbb{P}(Y=1) = 1/2$$

$\Rightarrow$

$$\mathbb{P}[Y \in \{0,1\}] = 1$$

$\Rightarrow$

Y is a discrete r.v. with support $\{0,1\}$ and

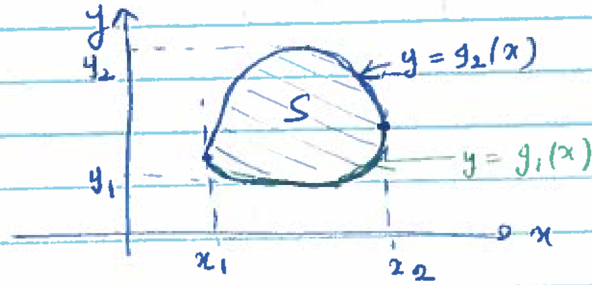
$P_Y(0) = P_Y(1) = 1/2$ //

7) X and Y are uniformly distributed on S if,

$$\mathbb{P}[(X,Y) \in B] = \int_B f(x,y)\,dx\,dy, \quad B \subseteq \mathbb{R}^2 \quad \text{— } ①'$$

where $f: \mathbb{R} \to \mathbb{R}_+$,

$$f(x,y) = \begin{cases} \frac{1}{Area(S)} & \text{if } (x,y) \in S \\ 0 & \text{if } (x,y) \notin S \end{cases}$$



Consider $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ — ①

for $x < x_1$ and $x > x_2$, $f(x,y) = 0 \Rightarrow f_X(x) = 0$.

for $x_1 \leq x \leq x_2 \Rightarrow$

$$f_X(x) = \int_{g_1(x)}^{g_2(x)} \frac{1}{Area(S)}\,dy$$

$$= \frac{g_2(x) - g_1(x)}{Area(S)}$$

$$f_X(x) = \begin{cases} \frac{g_2(x) - g_1(x)}{Area(S)}, & x_1 \leq x \leq x_2 \\ 0, & \text{otherwise} \end{cases} \quad \text{— } ②$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \quad (\text{from } ①' \text{ and } ①)$$

$$= \int_{\mathbb{R}^2} f(x,y)\,dx\,dy$$

$$= \int_S \frac{1}{Area(S)}\,dx\,dy$$

$$= \frac{1}{Area(S)} \Big(\underbrace{\int_S dx\,dy}_{Area(S)} \Big)$$

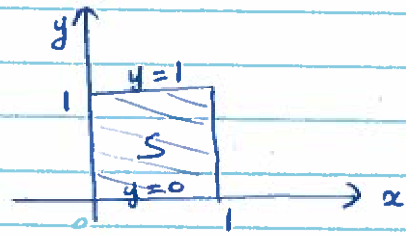
$\therefore \int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{Area(S)} \cdot Area(S) = 1$

$\Rightarrow f_X(x)$ is a valid density function.

$\therefore X$ is a continuous rv, with density $f_X(x)$.

Similarly, Y is also a continous rv, with a density function $f_Y(y) > 0$ for $y_1 \leq y \leq y_2$.

(7.a)



$\Rightarrow Area(S) = 1$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 \frac{1}{1}\,dy \quad , \quad 0 \leq x \leq 1$$

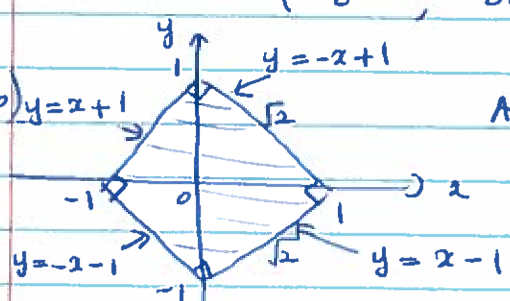
$$= 1$$

$$\therefore f_X(x) = \begin{cases} 1 & , \; 0 \leq x \leq 1 \\ 0 & , \; \text{otherwise.} \end{cases}$$

By the symmetry,

$$f_Y(y) = \begin{cases} 1 & , \; 0 \leq y \leq 1 \\ 0 & , \; \text{otherwise.} \end{cases}$$

(7.b)



$Area(S) = \sqrt{2} \times \sqrt{2} = 2.$

For $-1 \leq x \leq 0 \Rightarrow f_X(x) = \int_{-x-1}^{x+1} \frac{1}{2}\,dy = \frac{1}{2} \times 2(x+1) = x+1$

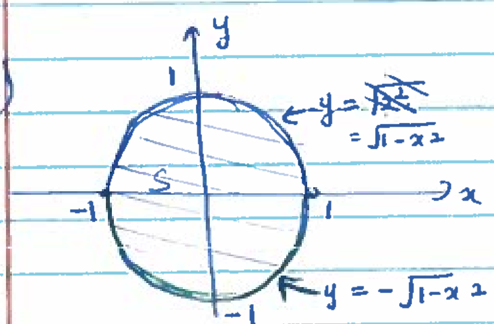
For $0 \le x \le 1 \Rightarrow f_X(x) = \int_{x-1}^{-x+1} \frac{1}{2}\, dy = (1-x)$

$\therefore f_X(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$ //

By the symmetry,

$f_Y(y) = \begin{cases} y+1, & -1 \le y \le 0 \\ 1-y, & 0 < y \le 1 \\ 0, & \text{otherwise.} \end{cases}$ //

(7.c)



$y^2 + x^2 = 1$

$y = \pm\sqrt{1-x^2}$

$\text{Area}(S) = \pi(1)^2 = \pi$

$\therefore$ for $-1 \le x \le 1$, $f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy$

$= \frac{2}{\pi}\sqrt{1-x^2}$

$\therefore f_X(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

By the symmetry, we can conclude that,

$f_Y(y) = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2}, & |y| \le 1 \\ 0, & |y| > 1 \end{cases}$

9) $f_X(x) = \begin{cases} 0 & , x < 1 \\ Cx^{-4} & , x \geq 1 \end{cases}$

$C > 0.$

a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \int_{1}^{\infty} Cx^{-4}\,dx = 1 \Rightarrow C = 3.$

So, $f_X(x) = \begin{cases} 3x^{-4}, & x \geq 1 \\ 0 & , x < 1 \end{cases}$

b) $\mathbb{P}[0.5 < X < 2] = \int_{0.5}^{2} f_X(x)\,dx = \int_{1}^{2} 3x^{-4}\,dx = \frac{7}{8}.$

$\mathbb{P}[2 < X < 4] = \int_{2}^{4} f_X(x)\,dx = \int_{2}^{4} 3x^{-4}\,dx = \frac{7}{64}.$

c) $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

So, $F_X(x) = 0$ if $x < 1$

If $x \geq 1$,
$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{1}^{x} 3x^{-4}\,dx = 1 - \frac{1}{x^3}.$$

So, $F_X(x) = \begin{cases} 1 - \frac{1}{x^3}, & x \geq 1 \\ 0 & , x < 1. \end{cases}$

d) $\mathbb{E}[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx, \quad n = 1, 2, \ldots$

$= 3\int_{1}^{\infty} x^{n-4}\,dx$

If $n = 3$, $\mathbb{E}[X^n] = 3\int_{1}^{\infty} \frac{dx}{x} = 3\left|\ln x\right|_{1}^{\infty} \to \infty.$

If $n \neq 3$, $\mathbb{E}[X^n] = 3 \cdot \left|\frac{x^{n-3}}{n-3}\right|_{1}^{\infty} = \frac{3}{n-3}\left|x^{n-3}\right|_{1}^{\infty} = \begin{cases} \frac{3}{3-n}, & \text{if } n < 3 \\ \infty & , \text{if } n > 3. \end{cases}$

So, $\mathbb{E}[X^n] = \begin{cases} \frac{3}{3-n} & , n = 1,2 \\ \infty & , n = 3,4,\ldots \end{cases}$

10) $f_X(x) = \begin{cases} 0 & , x < 1 \\ \frac{1}{2}x^{-3/2} & , x \geq 1 \end{cases}$

a) $\mathbb{P}[X > 10] = \int_{10}^{\infty} f_X(x)\,dx = \int_{10}^{\infty} \frac{1}{2}x^{-3/2}\,dx = \frac{1}{\sqrt{10}}$.

b) $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

So, $F_X(x) = 0$, if $x < 1$

If $x \geq 1$,
$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{1}^{x} \frac{1}{2}x^{-3/2}\,dx = 1 - \frac{1}{\sqrt{x}}.$$

So, $F_X(x) = \begin{cases} 1 - \frac{1}{\sqrt{x}} & , x \geq 1 \\ 0 & , x < 1 \end{cases}$

c) $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$= \int_{1}^{\infty} x \cdot \frac{1}{2}x^{-3/2}\,dx$$

$$= \frac{1}{2}\int_{1}^{\infty} x^{-1/2}\,dx$$

$$= \frac{1}{2}\left|\frac{x^{1/2}}{1/2}\right|_{1}^{\infty} \rightarrow \infty.$$

d) $\mathbb{E}[X^{1/4}] = \int_{-\infty}^{\infty} x^{1/4} \cdot f_X(x)\,dx$

$= \int_{1}^{\infty} x^{1/4} \cdot \frac{1}{2} x^{-3/2}\,dx$

$= \frac{1}{2}\int_{1}^{\infty} x^{-5/4}\,dx$

$= \frac{1}{2}\left| \frac{x^{-1/4}}{-1/4} \right|_{1}^{\infty}$

$= -2(0-1)$

$= 2.$