ENGINEERING PROBABILITY

HOMEWORK # 11:
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Please work out the **ten** (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem **1.55** (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). **Show** work and **explain** reasoning.

1.

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right), \quad x \in \mathbb{R}.$$

1.a. Show that f is a probability density function (known as the Cauchy probability density function).

1.b. Find the corresponding cumulative distribution function $F : \mathbb{R} \to [0,1]$ given by

$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad x \in \mathbb{R}.$$

1.c. Can you evaluate the first moment of this probability distribution? Explain your answer.

2.

The standard Gaussian probability density function $\phi : \mathbb{R} \to \mathbb{R}_+$ is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

Use the fact that $\phi : \mathbb{R} \to \mathbb{R}_+$ is a probability density function to compute the integrals

$$I(a,b) = \int_{\mathbb{R}} e^{-(a^2x^2+bx)}dx$$

with $a > 0$ and b in $\mathbb{R}$. **HINT:** Complete the square in the quadratic form $a^2x^2 + bx$!

3.

The rv $X : \Omega \to \mathbb{R}$ has cumulative distribution function given by

$$\mathbb{P}\left[X \le x\right] = \begin{cases} 0 & \text{if } x \le 0 \\ \\ \frac{x}{1+x} & \text{if } x > 0. \end{cases}$$

3.a. Find the probability density function of the rv X.

3.b. Compute $\mathbb{P}\left[2 < X < 3\right]$.

3.c. Evaluate the quantity

$$\mathbb{E}\left[(1+X)^2 e^{-2X}\right].$$

4.

Consider rvs $X_1, \ldots, X_d : \Omega \to \mathbb{R}$ all defined on some probability triple $(\Omega, \mathcal{A}, \mathbb{P})$. Recall that we say that the rvs $X_1, \ldots, X_d$ are mutually independent if the conditions

$$\mathbb{P}\left[X_1 \in B_1, \ldots, X_d \in B_d\right] = \prod_{i=1}^{d} \mathbb{P}\left[X_i \in B_i\right], \quad B_i \subseteq \mathbb{R},\ i = 1, \ldots, d$$

simultaneously hold. It was argued that this definition is equivalent to the following seemingly weaker conditions

$$\mathbb{P}\left[X_1 \le x_1, \ldots, X_d \le x_d\right] = \prod_{i=1}^{d} \mathbb{P}\left[X_i \le x_i\right], \quad x_i \in \mathbb{R},\ i = 1, \ldots, d$$

simultaneously hold.

4.a. If for each $i = 1, \ldots, d$, the rvs X_i is of continuous type with probability density function $f_i : \mathbb{R} \to \mathbb{R}_+$, then show that the mutual independence of the rvs $X_1, \ldots, X_d$ implies that the rv $(X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d$ is also of continuous type, and that its probability density function $f : \mathbb{R}^d \to \mathbb{R}_+$ is given by

$$f(x_1, \ldots, x_d) = \prod_{i=1}^{d} f_i(x_i), \qquad \begin{array}{c} x_i \in \mathbb{R} \\ i = 1, \ldots, n. \end{array}$$

4.b. Conversely, assume that the rv $(X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d$ is of continuous type, and that its probability density function $f : \mathbb{R}^d \to \mathbb{R}_+$ is of the form

$$f(x_1, \ldots, x_d) = \prod_{i=1}^{d} f_i(x_i), \qquad \begin{array}{c} x_i \in \mathbb{R} \\ i = 1, \ldots, n \end{array}$$

for some mappings $f_1, \ldots, f_d : \mathbb{R}^d \to \mathbb{R}_+$. Under these assumptions, show that the rvs $X_1, \ldots, X_d$ are mutually independent and that each of these rvs is of continuous type with probability density functions $f_1, \ldots, f_d : \mathbb{R} \to \mathbb{R}_+$.

5.

Problem **3.5** (BT)

6.

Problem **3.6** (BT)

7.

Problem **3.7** (BT)

8.

Problem **3.8** (BT)

9.

You are told that the rv $X : \Omega \to \mathbb{R}$ is a Gaussian rv with parameters $m \in \mathbb{R}$ and $\sigma^2 > 0$.

9.a. For $a \neq 0$ and b arbitrary, show that the rv $Y = aX + b$ is also a Gaussian rv and find its probability density function – The case > 0 was already discussed in class.

9.b. Find x in $\mathbb{R}$ such that

$$\begin{array}{ll} \mathbb{P}\left[X \geq x\right] = .95 & \text{if } m = 1 \text{ and } \sigma = 2 \\ \mathbb{P}\left[X \geq x\right] = .0068 & \text{if } m = 6 \text{ and } \sigma = 3 \\ \mathbb{P}\left[X > x\right] = 3\mathbb{P}\left[X \leq x\right] & \text{if } m = 2 \text{ and } \sigma = 3 \\ \mathbb{P}\left[|X| \geq x\right] = .0076 & \text{if } m = 0 \text{ and } \sigma = 1 \end{array}$$

10.

The two-dimensional rv $(X, Y) : \Omega \to \mathbb{R}^2$ is uniformly distributed over the region $S \subseteq \mathbb{R}^2$. Define the rv $R = \sqrt{X^2 + Y^2}$ which gives the distance from the origin $(0, 0)$ to the random point (X, Y).

10.a. Assume first that the region S is the the circular region

$$S = \left\{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 \leq R^2\right\}.$$

Find the cumulative distribution function F_R of the rv R and its probbability distribution function f_R (if applicable). Is it a well-known distribution?

10.b. Repeat Part **a** when S is the circle

$$S = \left\{(x, y) \in \mathbb{R}^2 : \ x^2 + y^2 = R^2\right\}.$$