ENEE 324-01 / SPRING 2018

HOMEWORK # 11

SOLUTIONS

① $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right)$, $x \in \mathbb{R}$

(i) Note that $f(x) \geq 0$, $\forall x \in \mathbb{R}$

(ii) $$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{\pi}\left(\tan^{-1}(x)\right)_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left(\pi/2 - (-\pi/2)\right)$$

$$= 1$$

From ① and (ii), it is evident that $f(x)$ is a valid pdf.

(b) $$F(x) = \int_{-\infty}^{x} f(t)\, dt, \quad x \in \mathbb{R}$$

$$= \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+t^2}\right) dt$$

$$= \frac{1}{\pi} \tan^{-1}(t)\Big|_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x) + \pi/2\right]$$

(c) Consider the absolutely summability criterion for $E(X)$.

$$E[|X|] = \int_{-\infty}^{\infty} |x| f(x)\, dx$$

$$= \int_{-\infty}^{\infty} |x| \frac{1}{\pi}\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{x}{1+x^2}\, dx$$

$$\mathbb{E}[|X|] = \frac{1}{\pi} \ln|1+x^2| \Big|_0^{\infty}$$

$$= \infty \quad (\text{not finite})$$

Further note that $\int_0^{\infty} \frac{x}{1+x^2}\,dx = \infty$ and $\int_{-\infty}^{0} \frac{x}{1+x^2}\,dx = -\infty$

Hence, $\mathbb{E}[X]$ is not defined for this probability distribution.

(2)

$$I(a,b) = \int_{\mathbb{R}} e^{-(a^2x^2+bx)}\,dx$$

$$= \int_{\mathbb{R}} \exp\left[-a^2\left(x^2+\frac{bx}{a^2}+\frac{b^2}{4a^4}-\frac{b^2}{4a^4}\right)\right]dx$$

$$= \int_{\mathbb{R}} \exp\left[-a^2\left(\left(x+\frac{b}{2a^2}\right)^2-\frac{b^2}{4a^4}\right)\right]dx$$

$$= \int_{\mathbb{R}} \exp\left[\frac{-\left(x+\frac{b}{2a^2}\right)^2}{2\left(\frac{1}{\sqrt{2}a}\right)^2}\right]\cdot\exp\left[\frac{b^2}{4a^2}\right]dx$$

$$= e^{\left(\frac{b^2}{4a^2}\right)}\int_{\mathbb{R}} \exp\left\{\frac{-\left(x+\frac{b}{2a^2}\right)^2}{2\left(\frac{1}{\sqrt{2}a}\right)^2}\right\}dx$$

$$= \sqrt{\frac{\pi}{a^2}}\, e^{\left(\frac{b^2}{4a^2}\right)}\int_{\mathbb{R}} \underbrace{\frac{1}{\sqrt{2\pi\left(\frac{1}{\sqrt{2}a}\right)^2}}\exp\left\{\frac{-\left(x+\frac{b}{2a^2}\right)^2}{2\left(\frac{1}{\sqrt{2}a}\right)^2}\right\}}_{\sim N\left(-\frac{b}{2a^2},\ \left(\frac{1}{\sqrt{2}a}\right)^2\right)}dx$$

$$= \frac{\sqrt{\pi}}{|a|}\, e^{b^2/(4a^2)}$$

//

$$\left(\because \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1, \text{ when } \sim N(\mu,\sigma^2)\right)$$

③ $\mathbb{P}[X \le x] = F_X(x) = \begin{cases} 0 & , x \le 0 \\ \frac{x}{1+x} & , x > 0 \end{cases}$

$f_X(x) = \frac{d F_X(x)}{dx}$

$= \begin{cases} 0 & , x \le 0 \\ \frac{(1+x) - x}{(1+x)^2} = \frac{1}{(1+x)^2} & , x > 0 \end{cases}$

$= \begin{cases} 0 & , x \le 0 \\ \frac{1}{(1+x)^2} & , x > 0 \end{cases}$ //

(b) $\mathbb{P}[2 < X < 3] = \mathbb{P}[2 < X \le 3]$ $\left[\because \mathbb{P}(X=3) = 0, \ X \text{ is continous}\right]$

$= F_X(3) - F_X(2)$

$= \frac{3}{1+3} - \frac{2}{1+2}$

$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$ //

(c) $\mathbb{E}\left[(1+X)^2 e^{-2X}\right] = \int_0^\infty (1+x)^2 e^{-2x} f_X(x) \, dx$

$= \int_0^\infty (1+x)^2 e^{-2x} \left(\frac{1}{1+x}\right)^2 dx$

$= \left(\frac{e^{-2x}}{-2}\right)_0^\infty$

$= 1/2$ //

4) In the probability triple $(\Omega, \mathcal{A}, \mathbb{P})$, consider the rvs

$$X_1, \ldots, X_d : \Omega \to \mathbb{R}$$

These rvs are mutually independent if

i) $\mathbb{P}[X_1 \in B_1, \ldots, X_d \in B_d] = \prod_{i=1}^{d} \mathbb{P}[X_i \in B_i]$, $B_i \in \mathcal{R}$, $i = 1, \ldots, d$

OR

ii) $\mathbb{P}[X_1 \le x_1, \ldots, X_d \le x_d] = \prod_{i=1}^{d} \mathbb{P}[X_i \le x_i]$, $x_i \in \mathbb{R}$, $i = 1, \ldots, d$ hold.

a) For each $i = 1, \ldots, d$,

X_i is continuous rv with density $f^{\underline{n}}$ (pdf) $f_i : \mathbb{R} \to \mathbb{R}_+$

We have to show,

mutual independence of $X_1, \ldots, X_d$

$\Rightarrow$ $(X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d$ is continuous with density $f^{\underline{n}}$ $f : \mathbb{R}^d \to \mathbb{R}_+$

given by $f(x_1, \ldots, x_d) = \prod_{i=1} f_i(x_i)$, $\forall x_i \in \mathbb{R}$
$i = 1, \ldots, d$.

Proof:

$X_1, \ldots, X_d$ are mutually independent

$\Rightarrow$ $\mathbb{P}[X_1 \le x_1, \ldots, X_d \le x_d] = \prod_{i=1}^{d} \mathbb{P}[X_i \le x_i]$, $\forall x_i \in \mathbb{R}$, $i = 1, \ldots, d$

$\Rightarrow$ $F_{X_1, \ldots, X_d}(x_1, \ldots, x_d) = \prod_{i=1}^{d} F_{X_i}(x_i)$ ——— (i)

Since X_i's are all continuous with pdf $f_i(x_i) = \frac{\partial}{\partial x_i} F_{X_i}(x_i)$,

$F_{X_i}(x_i)$, $\forall x_i$, $i = 1, \ldots, d$ are all differentiable

$\Rightarrow \prod_{i=1}^{d} F_{X_i}(x_i)$ is differentiable

$\Rightarrow$ $F_{X_1, \ldots, X_d}(x_1, \ldots, x_d)$ is differentiable.

[Note that, $F_{X_1, \ldots, X_d}(x_1, \ldots, x_d)$ is the distribution $f^{\underline{n}}$ (cdf) of the joint rv $(X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d$.]

$$\therefore \frac{\partial^d}{\partial x_1 \ldots \partial x_d} F_{X_1, \ldots, X_d}(x_1, \ldots, x_d) = \frac{\partial^d}{\partial x_1 \ldots \partial x_d} \prod_{i=1}^{d} F_{X_i}(x_i) \quad \text{[from (i)]}$$

$$= \frac{\partial^{d-1}}{\partial x_1 \ldots \partial x_{d-1}} \left\{ \frac{\partial}{\partial x_d} \left(F_{X_1}(x_1) \ldots F_{X_d}(x_d) \right) \right\}$$

$$= \frac{\partial^{d-1}}{\partial x_1 \ldots \partial x_{d-1}} \left\{ F_{X_1}(x_1) \ldots F_{X_{d-1}}(x_{d-1}) \cdot f_d(x_d) \right\}$$

$$= \frac{\partial^{d-2}}{\partial x_1, \dots \partial x_{d-2}} \left\{ \frac{\partial}{\partial x_{d-1}} \{F_{x_1}(x_1) \dots F_{x_{d-1}}(x_{d-1}) . f_d(x_d)\} \right\}$$

$$= \frac{\partial^{d-2}}{\partial x_1 \dots \partial x_{d-2}} \{ F_{x_1}(x_1) \dots F_{x_{d-2}}(x_{d-2}) f_{d-1}(x_{d-1}) f_d(x_d) \}$$

$$\vdots$$

$$= f_1(x_1) \dots . f_d(x_d)$$

$$= \prod_{i=1}^{d} f_i(x_i)$$

Since $f_i : \mathbb{R} \to \mathbb{R}_+^* \; \forall i = 1, \dots, d$,

$$\prod_{i=1}^{d} f_i(x_i) \text{ is non-negative } \forall x_i \in \mathbb{R}, i = 1, \dots, d$$

Hence, $(X_1, \dots, X_d)$ is continuous rv with pdf given by

$$f(x_1, \dots, x_d) = \prod_{i=1}^{d} f_i(x_i) . \qquad \text{(Proved)}$$

[Recall the def^{n} of a continuous rv here.

A rv X is continuous, if $\exists$ a mapping $f_X : \mathbb{R}^d \to \mathbb{R}_+$ (called pdf) s.t.

$$F_X(x_1, \dots, x_d) = \int_{-\infty}^{x_1} \dots \int_{-\infty}^{x_d} f_X(\xi_1, \dots, \xi_d) \, d\xi_1 \dots d\xi_d \qquad (B)$$

$$\text{i.e., } f_X(x_1, \dots, x_d) = \frac{\partial^d}{\partial x_1 \dots \partial x_d} F_X(x_1, \dots, x_d) \qquad (A)]$$

b) $(X_1, \dots, X_d) : \Omega \to \mathbb{R}^d$ is a continuous rv with pdf $f : \mathbb{R}^d \to \mathbb{R}_+$ given by

$$f(x_1, \dots, x_d) = \prod_{i=1}^{d} f_i(x_i) \;, \forall x_i \in \mathbb{R}, i = 1, \dots, d$$

for some mappings $f_1, \dots, f_d : \mathbb{R} \to \mathbb{R}_+$.

We have to show that,

the rvs $X_1, \dots, X_d$ are mutually independent

and each X_i is continuous with pdf of X_i being $f_i : \mathbb{R} \to \mathbb{R}_+$, $i = 1, \dots, d$.

Proof:

The rv $(X_1, \ldots, X_d) : \Omega \to \mathbb{R}^d$ is continuous with pdf

$$f(x_1, \ldots, x_d) = \prod_{i=1}^{d} f_i(x_i), \quad \forall x_i \in \mathbb{R}, \; i = 1, \ldots, d$$

for some mappings $f_i : \mathbb{R} \to \mathbb{R}_+$, $i = 1, \ldots, d$. ——— (ii)

Integrating both sides w.r.t. $s_1, \ldots, s_d$,

$$\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(s_1, \ldots, s_d)\, ds_1 \ldots ds_d = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \prod_{i=1}^{d} f_i(s_i)\, ds_1 \ldots ds_d, \quad \forall x_i \in \mathbb{R}, \; i = 1, \ldots, d$$

$$\Rightarrow F_{X_1, \ldots, X_d}(x_1, \ldots, x_d) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} f(s_1, \ldots, s_d)\, ds_1 \ldots ds_d \quad \text{——— (iii)}$$

[by def^n (B) of the continuous rv $(X_1, \ldots, X_d)$]

$$= \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \prod_{i=1}^{d} f_i(s_i)\, ds_1 \ldots ds_d$$

$$= \int_{-\infty}^{x_1} f_1(s_1)\, ds_1 \cdots \int_{-\infty}^{x_d} f_d(s_d)\, ds_d$$

$$= \prod_{i=1}^{d} \int_{-\infty}^{x_i} f_i(s_i)\, ds_i \quad \text{——— (iv)}$$

Now, $F_{X_i}(x_i) = \mathbb{P}[X_i \leq x_i]$, $\forall x_i \in \mathbb{R}$, $i = 1, \ldots, d$

$$= \mathbb{P}[X_1 < \infty, \ldots, X_i \leq x_i, \ldots, X_d < \infty]$$

$$= F_{X_1, \ldots, X_d}(\infty, \ldots, x_i, \ldots \infty)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{x_i} \cdots \int_{-\infty}^{\infty} f(s_1, \ldots, s_d)\, ds_1 \ldots ds_d \quad \text{[from (iii)]}$$

$$= \int_{-\infty}^{x_i} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(s_1, \ldots, s_d)\, ds_1 \ldots ds_{i-1}\, ds_{i+1} \ldots ds_d \right] ds_i$$

The RHS is differentiable w.r.t. x_i

$\Rightarrow$ The LHS is differentiable w.r.t. x_i

$$\therefore \frac{\partial}{\partial x_i} F_{X_i}(x_i) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{(d-1)} f(\xi_1, \ldots, x_i, \ldots \xi_d)\, \underbrace{d\xi_1 \ldots d\xi_{i-1}\, d\xi_{i+1} \ldots d\xi_d}_{(d-1)}$$

Since $f : \mathbb{R}^d \to \mathbb{R}_+^+$,

the above integral is non-negative

$\Rightarrow \frac{\partial}{\partial x_i} F_{X_i}(x_i)$ is non-negative, $\forall x_i \in \mathbb{R}$, $i = 1, \ldots, d$

$\Rightarrow X_i$ is continuous with pdf

$$f_{X_i}(x_i) = \frac{\partial}{\partial x_i} F_{X_i}(x_i) \quad [\text{by def}^{n} \ (A)], \ \forall i = 1, \ldots, d.$$

So we have for each i,

$$f_{X_i}(x_i) = \frac{\partial}{\partial x_i} F_{X_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\xi_1, \ldots, x_i, \ldots, \xi_d)\, d\xi_1 \ldots d\xi_{i-1}\, d\xi_{i+1} \ldots d\xi_d$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(\xi_1) \ldots f_i(x_i) \ldots f_d(\xi_d)\, d\xi_1 \ldots d\xi_{i-1}\, d\xi_{i+1} \ldots d\xi_d$$

$$= \prod_{j \neq i} \int_{-\infty}^{\infty} f_j(\xi_j)\, d\xi_j \cdot f_i(x_i) \qquad \text{——— (v)}$$

Now, $F_{X_1, \ldots, X_d}(\infty, \ldots, \infty) = 1$

$$\Rightarrow \prod_{i=1}^{d} \int_{-\infty}^{\infty} f_i(\xi_i)\, d\xi_i = 1 \qquad \text{——— (vi)}$$

pdf of a continuous r.v X_i is unique and $\int_{-\infty}^{\infty} f_{X_i}(x_i) = 1$

Then, $f_{X_i}(x_i) = f_i(x_i)$ $\forall i$ satisfy (vi).

So we can assume

$$f_{X_i}(x_i) = f_i(x_i). \quad \forall x_i \in \mathbb{R}, \ i = 1, \ldots, d$$

i.e., each X_i is continuous with pdf $f_i : \mathbb{R} \to \mathbb{R}_+$. (Proved)

To show mutual independence:

(iv) $\Rightarrow F_{X_1,\dots,X_d}(x_1,\dots,x_d) = \prod_{i=1}^{d} \int_{-\infty}^{x_i} f_{X_i}(\xi_i)\, d\xi_i \quad [\because f_{X_i}(x_i) = f_i(x_i)\ \forall x_i \in \mathbb{R}]$

$$\Rightarrow \mathbb{P}[X_1 \le x_1, \dots X_d \le x_d] = F_{X_1,\dots,X_d}(x_1,\dots,x_d)$$

$$= \prod_{i=1}^{d} \int_{-\infty}^{x_i} f_{X_i}(\xi_i)\, d\xi_i$$

$$= \prod_{i=1}^{d} \mathbb{P}[X_i \le x_i] \quad , \forall x_i \in \mathbb{R},\ i=1,\dots,d$$

Hence, $X_1,\dots,X_d$ are mutually independent by def^n (ii).

(Proved)

[In (a) we have shown,

X_i's are continuous with pdf f_i, for each i, and mutually independent

$\Rightarrow$ their joint rv $(X_1,\dots,X_d)$ is also continuous with pdf

$$f(x_1,\dots,x_d) = \prod_{i=1} f_i(x_i) \quad , \forall x_i \in \mathbb{R},\ i=1,\dots,d$$

In (b) we have shown the converse of it:

$(X_1,\dots,X_d)$ is continuous with pdf

$$f(x_1,\dots,x_d) = \prod_{i=1} f_i(x_i) \quad \forall x_i \in \mathbb{R},\ i=1,\dots,d$$

$\Rightarrow$ X_i's are continuous with pdf f_i and mutually independent

This means the statements are equivalent, i.e.,

"X_i's are continuous with pdf f_i and mutually independent" if and only if "their joint rv $(X_1,\dots,X_d)$ is continuous with pdf

$$f(x_1,\dots,x_d) = \prod_{i=1} f_i(x_i) \quad \forall x_i \in \mathbb{R},\ i=1,\dots,d."$$]

5)

6) See solution manual.

7)

8)

9) $X \sim N(m, \sigma^2)$

a) Done in Recitation 10, problem 2.(a)

b)

i) $m=1, \sigma=2$.

$\mathbb{P}[X \geq x] = 1 - \mathbb{P}[X \leq x]$ [X is continuous rv $\Rightarrow \mathbb{P}[X<x] = \mathbb{P}[X \leq x]$]

$= 1 - \mathbb{P}[m + \sigma Y \leq x]$ [If $X \sim N(m, \sigma^2)$, $X = \sigma Y + m$, then $Y \sim N(0,1)$, see Recitation 9]

$= 1 - \mathbb{P}[Y \leq \frac{x-m}{\sigma}]$

$= 1 - \Phi\left(\frac{x-m}{\sigma}\right)$ [$\Phi(\cdot)$ is the cdf of standard normal rv $Y \sim N(0,1)$]

$\mathbb{P}[X \geq x] = 0.95$

$\Rightarrow \Phi\left(\frac{x-m}{\sigma}\right) = 0.05$

$= \Phi(-1.64)$ [from the Standard Normal table]

$\Rightarrow \frac{x-m}{\sigma} = -1.64$

$\Rightarrow x = -1.64\sigma + m$

$\therefore x = -2.28$

ii) $m=6, \sigma=3$

Proceeding in same way as in (i),

$\Phi\left(\frac{x-m}{\sigma}\right) = 1 - 0.0068 = 0.9932$

$= \Phi(2.47)$ [from the Standard Normal table]

$\Rightarrow \frac{x-m}{\sigma} = 2.47$

$\Rightarrow x = 2.47\sigma + m$

$\therefore x = 13.41$

iii) $\mathbb{P}[X > x] = 3\,\mathbb{P}[X \leq x]$

$m = 2, \sigma = 3$

So, $1 - \mathbb{P}[X \leq x] = 3\,\mathbb{P}[X \leq x]$

$\Rightarrow \mathbb{P}[X \leq x] = 0.25$

$\Rightarrow \Phi\left(\frac{x-m}{\sigma}\right) = 0.25$

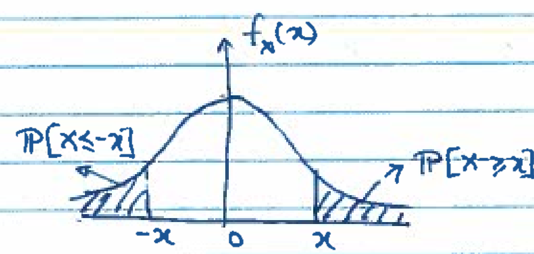
$= \Phi(-0.67)$

$\Rightarrow x = -0.67\sigma + m$

$\therefore x = -0.01$

iv) $m = 0, \sigma = 1$

$\therefore X \sim N(0,1)$



$\mathbb{P}[|X| \geq x] =$

$= \mathbb{P}[\{X \geq x\} \cup \{X \leq -x\}]$

$= \mathbb{P}[X \geq x] + \mathbb{P}[X \leq -x]$

$= 2 \cdot \mathbb{P}[X \leq -x]$ [by Symmetry of Standard Normal r.v.]

$= 2\Phi(-x)$

$\therefore \mathbb{P}[|X| \geq x] = 0.0076 \Rightarrow 2\Phi(-x) = 0.0076$

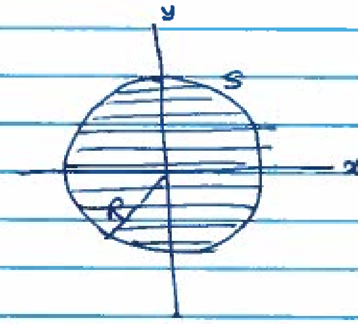
$\Rightarrow \Phi(-x) = 0.0038 = \Phi(-2.67)$

$\therefore x = 2.67.$

10) $(X,Y):\Omega\to\mathbb{R}^2$ is uniformly distributed over a region $S\subseteq\mathbb{R}^2$.

$R=\sqrt{X^2+Y^2}$

a) $S=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le R^2\}$



$F_R(r)=\mathbb{P}[R\le r]$

$=\mathbb{P}[\sqrt{X^2+Y^2}\le r]$

So, $F_R(r)=0$, if $r<0$ $\left[\text{Since, } \mathbb{P}[\sqrt{X^2+Y^2}<0]=0\right]$

If $0\le r\le R$,

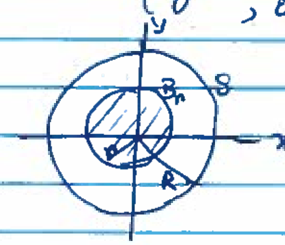
$$F_R(r)=\iint_{B_r} f_{X,Y}(x,y)\,dx\,dy \quad \text{, where } B_r=\{(x,y)\in\mathbb{R}^2 : \sqrt{x^2+y^2}\le r\}$$

$$=\iint_{B_r}\frac{1}{\text{Area}(S)}\,dx\,dy \quad [\because B_r\subseteq S]$$

$\left[\because (X,Y)\text{ is uniformly distributed over } S=\{(x,y): x^2+y^2\le R^2\} \Rightarrow f_{X,Y}(x,y)=\begin{cases}\frac{1}{\text{Area}(S)} & \text{if } (x,y)\in S\\ 0 & \text{, o/w}\end{cases}\right]$

$$=\iint_{B_r}\frac{1}{\pi R^2}\,dx\,dy$$

$$=\frac{1}{\pi R^2}\iint_{B_r}dx\,dy$$



$$=\frac{1}{\pi R^2}\cdot\text{Area}(B_r)$$

$$=\frac{\pi r^2}{\pi R^2}$$

$$=\frac{r^2}{R^2}$$

If $r\ge R$,

$F_R(r)=1$. $[\because r\ge R\Rightarrow S\subseteq B_r \Rightarrow \mathbb{P}[(X,Y)\in S]\le\mathbb{P}[(X,Y)\in B_r]$

but $\mathbb{P}[(X,Y)\in S]=1$

$\therefore \mathbb{P}[(X,Y)\in B_r]\ge 1\Rightarrow \mathbb{P}[(X,Y)\in B_r]=1$ $]$

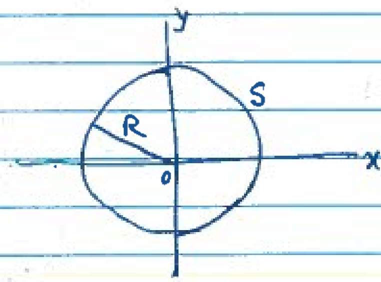
So, $F_R(r)=\begin{cases}0 & , r\le 0\\ \frac{r^2}{R^2} & , 0\le r\le R\\ 1 & , r\ge R\end{cases}$

So, $F_R(r)$ is continuous w.r.t. r.

$$\therefore \; f_R(r) = \frac{d}{dr}F_R(r) = \begin{cases} 0, & r \le 0 \\ \frac{2r}{R^2}, & 0 \le r \le R \\ 0, & r > R \end{cases} = \begin{cases} \frac{2r}{R^2}, & 0 \le r \le R \\ 0, & o/w \end{cases}$$

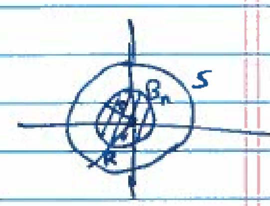
This is a triangular distribution with lower limit 0, upper limit R and mode R

b) $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = R^2\}$



(X,Y) is uniformly distributed on the circle.

$$\Rightarrow f_{X,Y}(x,y) = \begin{cases} \frac{1}{2\pi R}, & \text{if } x^2+y^2 = R^2 \\ 0, & o/w \end{cases}$$



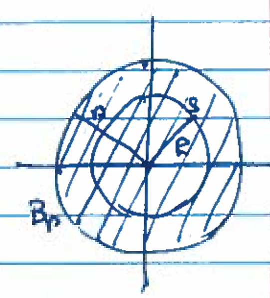
If $r < R$,

$$F_R(r) = \mathbb{P}[\sqrt{X^2+Y^2} < r]$$

$$= 0 \quad [\because \sqrt{X^2+Y^2} < r \Rightarrow \sqrt{X^2+Y^2} < R \text{ (as } r < R)$$
$$\text{So, } \mathbb{P}[\sqrt{X^2+Y^2} < r] \le \mathbb{P}[\sqrt{X^2+Y^2} < R] = 0 \,]$$

If $r \ge R$,

$$F_R(r) = \mathbb{P}[X^2+Y^2 \le r]$$



Since $B_r \subseteq S$,

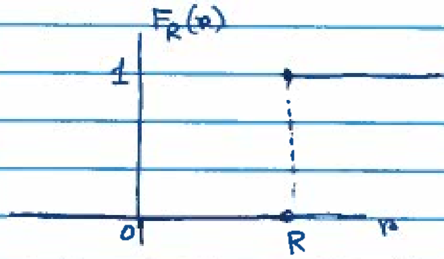
$$\mathbb{P}[(X,Y) \in B_r] \le \mathbb{P}[(X,Y) \in S] = 1$$

$$\Rightarrow \mathbb{P}[(X,Y) \in B_r] = 1$$

$$\Rightarrow \mathbb{P}[X^2+Y^2 \le r] = 1$$

$$\Rightarrow F_R(r) = 1.$$

So, $F_R(r) = \begin{cases} 0, & r < R \\ 1, & r \ge R \end{cases}$



Note that $F_R(r)$ is discontinuous with a jump at $r = R$.

This is an instance where 'function of continuous rvs may not be continuous'.