ENEE 324 - 01 / FALL 2018

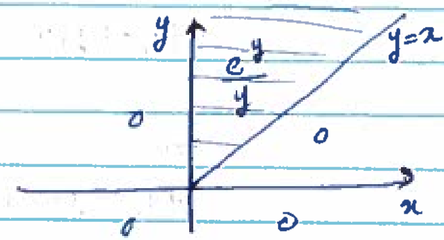
HOMEWORK # 12

SOLUTIONS

(1) $$f_{X,Y}(x,y) = \begin{cases} e^{-y}/y & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(1.a) Since X and Y are jointly continous, they will be marginally continous.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$



$$= \int_{x}^{\infty} \frac{e^{-y}}{y}\,dy$$

$$= E_1(x)\ ,\ x \geq 0 \qquad \left(\text{where } E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t}\,dt \right)$$

(This should be computed numerically, a closed form solution doesn't exist).

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_0^{y} \frac{e^{-y}}{y}\,dx$$

$$= e^{-y}\ ,\ y \geq 0$$

$$\therefore f_X(x) = \begin{cases} E_1(x), & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad f_Y(y) = \begin{cases} e^{-y}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

(b) $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$ for $\forall\, x, y \in \mathbb{R}$

Hence, $X \not\perp Y$ (X and Y are not independent).

(c) $$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \frac{e^{-y/y}}{e^{-y}}, \quad y>0 \text{ and } 0<x<y$$

$$= \frac{1}{y}, \quad 0<x<y$$

$\therefore$ For $y>0$,

$$f_{X|Y}(x|y) = \begin{cases} 1/y, & 0<x<y \\ 0, & x<0 \text{ or } x>y \end{cases}$$

(Note that $X|Y=y$ is uniform in $[0, y)$]

(1.d)

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

For $x>0$;

$$f_{Y|X}(y|x) = \begin{cases} \frac{e^{-y}}{y\,E_1(x)}, & y>x \\ 0, & y<x \end{cases}$$

It is not easy to derive a closed form solution for $f_{Y|X}(y|x)$!

$y>0$,

(1.e) $$E\left[X^3 \mid Y=y\right] = \int_{-\infty}^{\infty} f_{X|Y}(x|y) \cdot x^3 \, dx$$

$$= \int_0^y x^3/y \, dx$$

$$= \frac{1}{y}\left(\frac{x^4}{4}\right)_0^y$$

$$= y^3/4 \quad //, \quad y>0$$

2) (X,Y) is continuous with pdf

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{e^{-\frac{x}{y}} \cdot e^{-y}}{y}, & 0 < x, y < \infty \\ 0, & o/w \end{cases}$$

a) Since X and Y are jointly continuous, their marginal distribution is also continuous.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \begin{cases} \displaystyle\int_0^{\infty} e^{-\frac{x}{y}} \cdot e^{-y} \cdot \frac{1}{y}\,dy, & \text{if } 0 < x < \infty \\ 0, & o/w \end{cases}$$

The integral has to be evaluated numerically.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{0}^{\infty} e^{-\frac{x}{y}} \cdot e^{-y} \cdot \frac{1}{y}\,dx, \quad \text{if } 0 < y < \infty$$

$$= \frac{e^{-y}}{y} \int_0^{\infty} e^{-x/y}\,dy, \quad \text{if } 0 < y < \infty$$

$$= \frac{e^{-y}}{y} \left[\frac{e^{-x/y}}{-1/y} \right]_0^{\infty}, \quad \text{if } 0 < y < \infty$$

$$= e^{-y}, \quad \text{if } 0 < y < \infty$$

So, $$f_X(x) = \begin{cases} \displaystyle\int_0^{\infty} \frac{e^{-\frac{x}{y}} \cdot e^{-y}}{y}\,dy, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$$

$$f_Y(y) = \begin{cases} e^{-y}, & \text{if } y > 0 \\ 0, & \text{if } y \le 0 \end{cases}$$

b) $f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y) \quad \forall\, x, y \in \mathbb{R}$

Hence, X and Y are not independent.

At $x=1, y=1$, $f_{X,Y}(1,1) = e^{-1} \times e^{-1} = 0.135$

$f_X(1) \cdot f_Y(1) = 0.228 \times e^{-1} = 0.084 \neq f_{X,Y}(1,1)$.

c) $\mathbb{E}[X^2 | Y=y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\, dx$

Now,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{e^{-\frac{x}{y}} \cdot e^{-y} \cdot \frac{1}{y}}{e^{-y}}, \text{ if } 0 < x, y < \infty$$

$$= \begin{cases} \frac{e^{-\frac{x}{y}}}{y}, & \text{if } 0 < x, y < \infty \\ 0, & \text{o/w} \end{cases}$$

$$\therefore \mathbb{E}[X^2 | Y=y] = \int_0^{\infty} x^2 \cdot \frac{e^{-\frac{x}{y}}}{y}\, dx, \quad \text{if } y > 0$$

Use $\frac{x}{y} = t$ | $= \int_0^{\infty} y^2 t^2 e^{-t}\, dt$ [Note that y is fixed in this integral as conditioned on $\{Y=y\}$]

$$= y^2 \int_0^{\infty} t^2 e^{-t}\, dt$$

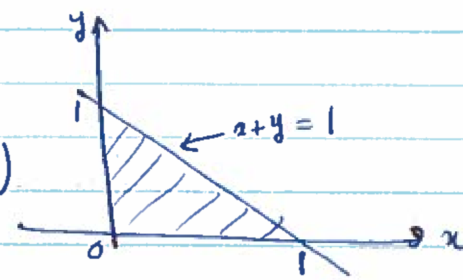
$$= y^2 \cdot \Gamma(3)$$ [where Gamma integral $\Gamma(n) = \int_0^{\infty} e^{-t} \cdot t^{n-1}\, dt$, $n \in \mathbb{N}$ $= (n-1)!$]

$$= 2y^2.$$

3) See Recitation 11 / Problem 2.

④ $X \perp Y$, $X \sim \exp(2)$, $Y \sim \exp(3)$

$A = [X + Y \leq 1]$



$$f_{XY}(x,y) = f_X(x)\, f_Y(y) \quad (X \perp Y)$$
$$= 2e^{-2x} \cdot 3e^{-3y}, \quad x \geq 0, y \geq 0$$

$$\mathbb{P}(A) = \mathbb{P}[X + Y \leq 1]$$
$$= \int_{x=0}^{1}\int_{y=0}^{1-x} f_{XY}(x,y)\, dy\, dx$$
$$= \int_0^1\int_0^{1-x} 6e^{-2x}e^{-3y}\, dy\, dx$$
$$= \int_0^1 6e^{-2x}\left(\int_0^{1-x} e^{-3y}\, dy\right) dx$$
$$= \int_0^1 6e^{-2x}\left(\frac{e^{-3y}}{-3}\right)_0^{1-x} dx$$
$$= \int_0^1 2e^{-2x}\left[1 - e^{-3(1-x)}\right] dx$$
$$= \left(-e^{-2x}\right)_0^1 - 2e^{-3}\left(e^{x}\right)_0^1$$
$$= -e^{-2} + 1 - 2e^{-3}\left[e^1 - 1\right]$$
$$= 1 - 3e^{-2} + 2e^{-3} \quad //$$

$$f_{XY|A}(x,y) = \begin{cases} \dfrac{f_{XY}(x,y)}{\mathbb{P}(A)}, & (x,y) \in A \\ 0, & \text{else} \end{cases}$$

$$= \begin{cases} \dfrac{6e^{-2x}e^{-3y}}{1 - 3e^{-2} + 2e^{-3}}, & x \geq 0, y \geq 0, x+y \leq 1 \\ 0, & \text{else} \end{cases} \quad //$$

Since the density function $f_{X,Y|A}(x,y)$ exists and is given above, the joint conditional distribution of the pair (X,Y) given A is of continous type.

5)

6)

7)

8)

9)

10)

See solution manual.