ENEE 324-01 / FALL 2018
HOMEWORK #13
SOLUTIONS

(1) Check Recitation 12 Notes

(2) X, Y, Z iid , $\mathbb{P}(X=1) = \mathbb{P}(X=2) = 1/2$

(2.a) $A = XYZ$

Given that $S_X = S_Y = S_Z = \{1, 2\}$, it is evident that $S_A = \{1, 2, 4, 8\}$

$$\mathbb{P}[A=1] = \mathbb{P}(XYZ=1)$$
$$= \mathbb{P}(X=1, Y=1, Z=1)$$
$$= \mathbb{P}(X=1)\,\mathbb{P}(Y=1)\,\mathbb{P}(Z=1) \quad (\because X \perp\!\!\!\perp Y \perp\!\!\!\perp Z)$$
$$= 1/8$$

$$\mathbb{P}[A=2] = \mathbb{P}(XZY=2)$$
$$= \mathbb{P}(X=1, Y=2, Z=1) + \mathbb{P}(X=1, Y=1, Z=2) + \mathbb{P}(X=2, Y=1, Z=1)$$
$$= \mathbb{P}(X=1)\mathbb{P}(Y=2)\mathbb{P}(Z=1) + \mathbb{P}(X=1)\mathbb{P}(Y=1)\mathbb{P}(Z=2) + \mathbb{P}(X=2)\mathbb{P}(Y=1)\mathbb{P}(Z=1)$$
$$= 3 \times 1/8 = 3/8$$

$$\mathbb{P}(A=4) = \mathbb{P}(X=1, Y=2, Z=2) + \mathbb{P}(X=2, Y=1, Z=2) + \mathbb{P}[X=2, Y=2, Z=1]$$
$$= 3 \times 1/8 = 3/8$$

$$\mathbb{P}(A=8) = \mathbb{P}(XZY=8)$$
$$= \mathbb{P}(X=2, Y=2, Z=2)$$
$$= \mathbb{P}(X=2)\cdot\mathbb{P}(Y=2)\cdot\mathbb{P}(Z=2)$$
$$= 1/8$$

(2b) Consider the different combinations of X, Y, Z and observe the value B takes.

$B = XY + YZ + ZX$

x	y	z	$\Rightarrow$	B
1	1	1		3
1	1	2		5
1	2	1		5
1	2	2		8
2	1	1		5
2	1	2		8
2	2	1		8
2	2	2		12

Since X, Y, Z are iid with $\mathbb{P}(X=1) = \mathbb{P}(X=2) = 1/2$,

$\mathbb{P}(X=x, Y=y, Z=z) = 1/8$ for $x, y, z \{1, 2\}$

Hence by observing the above tabulation,

$\mathbb{P}(B=3) = 1/8$
$\mathbb{P}(B=5) = 3/8$
$\mathbb{P}(B=8) = 3/8$
$\mathbb{P}(B=12) = 1/8$ //

④ $X \perp\!\!\!\perp Y$, $X, Y \sim N(0,1)$

$U = X$ and $V = \begin{cases} Y/X & \text{if } X \neq 0 \\ 0 & \text{if } X = 0 \end{cases}$

$$\mathbb{P}[U \le u, V \le v]$$
$$= \mathbb{P}[X \le u, V \le v]$$
$$= \mathbb{P}[X \le u, Y/X \le v, X \neq 0] + \underbrace{\mathbb{P}[X \le u, 0 \le v, X = 0]}_{\le \mathbb{P}(X=0)=0} \ge 0$$
$$= \mathbb{P}(X \le u, Y/X \le v, X \neq 0) + 0$$
$$= \mathbb{P}(X \le u, Y/X \le v, X < 0) + \mathbb{P}(X \le u, Y/X \le v, X > 0)$$
$$= \mathbb{P}(X \le u, X < 0, Y \ge vX) + \underline{\mathbb{P}(0 < X \le u, Y \le vX)} \text{ ①}$$

for $u < 0$:

$$① = \mathbb{P}(X \le u, Y \ge vX) + 0$$
$$= \int_{-\infty}^{\infty} \mathbb{P}[X \le u, Y \ge vX \mid X = x] f_X(x)\, dx$$
$$= \int_{-\infty}^{u} \mathbb{P}(Y \ge vx) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$
$$= \int_{-\infty}^{u} (1 - \Phi(vx)) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx, \text{ where } \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-y^2/2}\, dy$$

Clearly $\mathbb{P}[U \le u, V \le v]$ is differentiable and hence is of continuous type.

$$f_{U,V}(u,v) = \frac{\partial^2}{\partial v \partial u} \mathbb{P}[U \le u, V \le v]$$

$$\frac{\partial}{\partial u} \mathbb{P}(U \le u, V \le v) = (1 - \Phi(uv)) \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

$$\frac{\partial^2}{\partial v \partial u} \mathbb{P}(U \le u, V \le v) = -\frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{\partial}{\partial v}(\Phi(uv))$$
$$= -\frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{e^{-\frac{(vu)^2}{2}}}{\sqrt{2\pi}} u$$

$$= \frac{e^{-u^2/2} \, e^{-(uv)^2/2}}{2\pi} |u| \quad \text{——— (2)}$$

$$\left\{ \text{Where we use } \frac{\partial}{\partial v} \Phi(uv) = \frac{\partial}{\partial v}\left[\frac{1}{\sqrt{2\pi}} \int_{\infty}^{uv} e^{-t^2/2} dt\right] = \frac{e^{-\frac{(uv)^2}{2}} u}{\sqrt{2\pi}} \right.$$

<u>For $u > 0$</u>

$$(1) = \mathbb{P}\left(X \le u,\ X < 0,\ Y \ge vX\right) + \mathbb{P}\left[X \le u,\ Y \le vX,\ X > 0\right]$$

$$= \mathbb{P}\left(X < 0,\ Y \ge vX\right] + \mathbb{P}\left[0 < X \le u,\ Y \le vX\right]$$

$$= \int_{-\infty}^{\infty} \mathbb{P}\left(X < 0,\ Y \ge vX \mid X = x\right] f_X(x)\,dx + \int_{-\infty}^{\infty} \mathbb{P}\left[0 < X \le u,\ Y \le vX \mid X = x\right) f_X(x)\,d$$

$$= \int_{-\infty}^{0} \mathbb{P}\left[Y \ge vx\right] f_X(x)\,dx + \int_0^u \mathbb{P}\left[Y \le vx\right] f_X(x)\,dx$$

$$= \int_{-\infty}^{0} \left(1 - \Phi(vx)\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \int_0^u \Phi(vx) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= \int_{-\infty}^{0} \Phi(-vx) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \int_0^u \Phi(vx) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$= \int_0^{\infty} \Phi(vx) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx + \int_0^u \Phi(vx) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$f_{U,V}(u,v) = \frac{\partial^2}{\partial v \partial u} \mathbb{P}\left[U \le u,\ V \le v\right]$$

$$\frac{\partial}{\partial u} \mathbb{P}\left[U \le u,\ V \le v\right] = 0 + \frac{e^{-u^2/2}}{\sqrt{2\pi}} \Phi(uv)$$

$$\frac{\partial^2}{\partial v \partial u} \mathbb{P}\left[U \le u,\ V \le v\right] = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-(uv)^2/2} u$$

$$= \frac{e^{-u^2/2} \, e^{-(uv)^2/2}}{2\pi} |u| \quad \text{——— (3)}$$

from (2) and (3) ;

$$f_{U,V}(u,v) = \frac{e^{-u^2/2} \, e^{-(uv)^2/2}}{2\pi} |u| \quad //$$

$$f_{UV}(u,v) = \underbrace{\frac{e^{-u^2/2}}{\sqrt{2\pi}}}_{f_U(u)} \cdot \frac{e^{-\frac{(uv)^2}{2}}|u|}{\sqrt{2\pi}}$$

(Since $U = X$, $f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$)

$$\therefore f_{UV}(u,v) = f_U(u) \cdot \frac{e^{-\frac{(uv)^2}{2}}|u|}{\sqrt{2\pi}}$$

Clearly, $f_{UV}(u,v)$ cannot be equal to $f_U(u) \cdot f_V(v)$.
Hence, U and V are not independent.

Note 1: $V = Y/X$, ratio of two iid $N(0,1)$ variables is a Cauchy distribution.

Note 2: In the above calculations, we used differentiation under integral sign.

$$\frac{d}{d\alpha}\int_{f(\alpha)}^{g(\alpha)} \phi(x,\alpha)\,dx = \int_{f(\alpha)}^{g(\alpha)} \frac{\partial \phi(x,\alpha)}{\partial \alpha}\,dx + \phi(g(\alpha),\alpha)\frac{d\,g(\alpha)}{d\alpha} - \phi(f(\alpha),\alpha)\cdot\frac{df(\alpha)}{d\alpha}$$

In particular, for 'a' constant,

$$\frac{d}{d\alpha}\int_{a}^{g(\alpha)} \phi(x)\,dx = \phi(g(\alpha)) \cdot \frac{dg(\alpha)}{d\alpha} \quad //$$

5) $X \perp\!\!\!\perp Y$

$X \sim \mathrm{Unif}(0,1) \Rightarrow f_X(x) = 1$, if $0<x<1$

$Y \sim \mathrm{Unif}(0,1) \Rightarrow f_Y(y) = 1$, if $0<y<1$

$\therefore f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) = 1$, if $0<x<1,\ 0<y<1$

Note that $Y>X$ is an event and its complement is $Y \leq X$.

So we can write

$$\mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\cdot \mathbb{1}[\min(X,Y)\neq 0]\right]$$

$$= \mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\cdot \mathbb{1}[\min(X,Y)\neq 0] \,\middle|\, Y>X\right]\cdot \mathbb{P}[Y>X] + \mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\mathbb{1}[\min(X,Y)\neq 0] \,\middle|\, Y\leq X\right]\cdot \mathbb{P}[Y\leq X]$$

$$\mathbb{P}[Y=X] = \iint_B f_{X,Y}(x,y)\,dx\,dy, \quad \text{where } B = \{(x,y)\in\mathbb{R}^2 : x=y\}$$

$= 0.$ [The set B has zero area]

$$\therefore \mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\mathbb{1}[\min(X,Y)\neq 0]\right]$$

$$= \mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\mathbb{1}[\min(X,Y)\neq 0] \,\middle|\, Y>X\right]\cdot \mathbb{P}[Y>X] + \mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\mathbb{1}[\min(X,Y)\neq 0] \,\middle|\, Y<X\right]\cdot \mathbb{P}[Y<X]$$

Since the rvs X and Y are identically distributed and the above expectation is symmetric in X, Y.

$$\mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\mathbb{1}[\min(X,Y)\neq 0]\right]$$

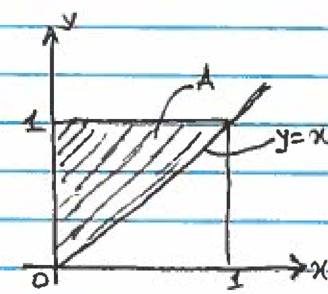
$$= 2\,\mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)}\mathbb{1}[\min(X,Y)\neq 0] \,\middle|\, Y>X\right]\cdot \mathbb{P}[Y>X] = 2\,\mathbb{E}\left[\frac{Y}{X}\cdot \mathbb{1}[X\neq 0] \,\middle|\, Y>X\right]\cdot \mathbb{P}[Y>X]$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y}{x}\cdot \mathbb{1}[x\neq 0]\, f_{X,Y|A}(x,y)\,dx\,dy \cdot \mathbb{P}[A] \quad [\text{where } A = [Y>X]]$$

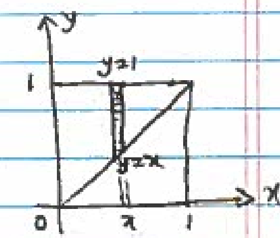
$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y}{x}\, f_{X,Y|A}(x,y)\,dx\,dy \cdot \mathbb{P}[A] \quad [\text{by continuity}]$$

Now, $f_{X,Y|A}(x,y) = \begin{cases} \dfrac{f_{X,Y}(x,y)}{\mathbb{P}[A]}, & \text{if } (x,y)\in A \\ 0, & \text{o/w} \end{cases}$

$= \begin{cases} \dfrac{1}{\mathbb{P}[A]}, & \text{if } 0<x<y<1 \\ 0, & \text{o/w} \end{cases}$



$\therefore \mathbb{E}\left[\frac{\max(X,Y)}{\min(X,Y)} \mathbb{1}[\min(X,Y)\neq 0]\right]$



$= 2\int_{x=0}^{1}\int_{y=x}^{1} \frac{y}{x}\cdot\frac{1}{\mathbb{P}[A]}\, dy\, dx \cdot \mathbb{P}[A]$

$= 2\int_{x=0}^{1} \frac{1}{x}\cdot \int_{y=x}^{1} y\, dy\; dx$

$= 2\int_{x=0}^{1} \frac{1}{x}\cdot \left[\frac{y^2}{2}\right]_x^1 dx$

$= \cancel{2}\int_{x=0}^{1} \frac{1-x^2}{\cancel{2}x}\, dx$

$= \left[\ln|x| - \frac{x^2}{2}\right]_0^1$

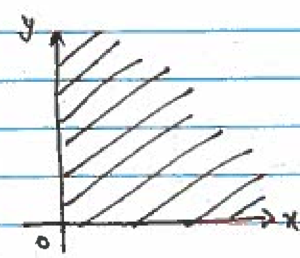
$= \infty.$

6> $X \perp\!\!\!\perp Y$

$X \sim exp(\lambda) \Rightarrow f_X(x) = \lambda e^{-\lambda x}$, if $x>0$

$Y \sim exp(\mu) \Rightarrow f_Y(y) = \mu e^{-\mu y}$, if $y>0$

$\therefore f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y) \quad \forall x,y$



$= \begin{cases} \lambda\mu e^{-\lambda x}\cdot e^{-\mu y}, & \text{if } x>0, y>0 \\ 0, & \text{o/w} \end{cases}$

$T = \begin{cases} \frac{Y}{X}, & \text{if } X\neq 0 \\ 0, & \text{if } X=0 \end{cases}$

a> Clearly $F_T(t) = \mathbb{P}[T\le t] = 0$ if $t\le 0$.

For $t>0$,

$F_T(t) = \mathbb{P}[T\le t] = \mathbb{P}[T\le t \mid X\neq 0]\cdot\mathbb{P}[X\neq 0] + \mathbb{P}[T\le t\mid X=0]\cdot\mathbb{P}[X=0]$

$= \mathbb{P}[T\le t\mid X\neq 0]\cdot\mathbb{P}[X\neq 0]$ $[\because \mathbb{P}[X=0]=0$ as X is continuous rv $]$

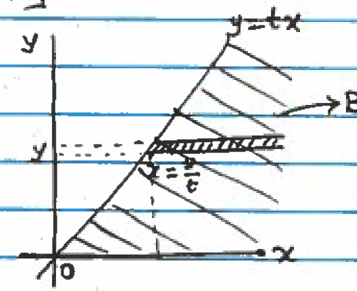
$= \mathbb{P}[\frac{Y}{X} \le t \mid X \ne 0] \cdot \mathbb{P}[X \ne 0]$

$= \mathbb{P}[\frac{Y}{X} \le t, X \ne 0]$

$= \mathbb{P}[Y \le tX, X \ne 0]$ $\quad [\because X \sim exp(\lambda) \Rightarrow \mathbb{P}[X>0]=1]$

$= \mathbb{P}[Y \le tX]$ $\quad [\because \mathbb{P}[X \ne 0] = 1]$

$= \iint_B f_{X,Y}(x,y)\,dx\,dy$



$= \int_{y=0}^{\infty} \int_{x=\frac{y}{t}}^{x=\infty} \lambda\mu e^{-\lambda x} . e^{-\mu y}\,dx\,dy$

$= \lambda\mu \int_{y=0}^{\infty} e^{-\mu y} \left[\frac{e^{-\lambda x}}{-\lambda}\right]_{x=\frac{y}{t}}^{x=\infty} dy$

$= \mu \int_{y=0}^{\infty} e^{-\mu y} . e^{-\lambda y/t}\,dy$

$= \mu \int_{y=0}^{\infty} e^{-(\mu + \frac{\lambda}{t})y} dy$

$= \frac{\mu}{\mu + \frac{\lambda}{t}}$

$= \frac{\mu t}{\mu t + \lambda}$

$\therefore \; F_T(t) = \begin{cases} \frac{\mu t}{\mu t + \lambda}, & \text{if } t \ge 0 \\ 0, & \text{if } t \le 0 \end{cases}$

b) Clearly $F_T(t)$ is a continuous f^n of t and differentiable w.r.t. t.

$\therefore \; f_T(t) = \frac{d}{dt} F_T(t) = \begin{cases} \frac{\mu\lambda}{(\mu t + \lambda)^2}, & t \ge 0 \\ 0, & o/w \end{cases}$

c) $\mathbb{E}[T] = \int_{-\infty}^{\infty} t f_T(t)\, dt$

$$= \int_0^{\infty} \frac{\mu\lambda}{(\mu t+\lambda)^2} \cdot t\, dt$$

$$= \int_{\lambda}^{\infty} \frac{\lambda(x-\lambda)}{\mu x^2}\, dx \qquad [\text{with } x = \mu t + \lambda]$$

$$= \frac{\lambda}{\mu}\int_{\lambda}^{\infty}\left(\frac{1}{x} - \frac{\lambda}{x^2}\right) dx$$

$$= \frac{\lambda}{\mu}\left[\ln|x| + \frac{\lambda}{x}\right]_{x=\lambda}^{\infty}$$

$$= \infty.$$

7) $R_1, \ldots, R_n$ are mutually independent and identically distributed ; $n \geq 3, n \in \mathbb{N}$

$R_k \sim \exp(\lambda)$, $\forall k, n \geq 3$.

a) $\mathbb{P}[R_1 > 2R_2] = \int_{-\infty}^{\infty} \mathbb{P}[R_1 > 2R_2 \mid R_2 = x]\, f_{R_2}(x)\, dx$

$$= \int_0^{\infty} \mathbb{P}[R_1 > 2R_2 \mid R_2 = x] \cdot \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} \mathbb{P}[R_1 > 2x \mid R_2 = x] \cdot \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} \mathbb{P}[R_1 > 2x] \cdot \lambda e^{-\lambda x}\, dx \qquad [\because R_1 \perp\!\!\!\perp R_2]$$

$$= \int_0^{\infty} e^{-\lambda \cdot 2x} \cdot \lambda e^{-\lambda x}\, dx$$

[Recall; if $X \sim \exp(\lambda)$, $F_X(x) = \mathbb{P}[X \leq x] = 1 - e^{-\lambda x}$ $\Rightarrow \mathbb{P}[X > x] = 1 - F_X(x) = e^{-\lambda x}$]

$$= \lambda \int_0^{\infty} e^{-3\lambda x}\, dx$$

$$= \frac{1}{3}.$$

b) $\mathbb{P}[R_1 > 2R_2 > 3R_3] = \int_{-\infty}^{\infty} \mathbb{P}[R_1 > 2R_2 > 3R_3 \mid R_2 = x]\, f_{R_2}(x)\,dx$

$= \int_0^{\infty} \mathbb{P}[R_1 > 2x > 3R_3 \mid R_2 = x]\, \lambda e^{-\lambda x}\,dx$

$= \int_0^{\infty} \mathbb{P}[R_1 > 2x, R_3 < \frac{2}{3}x]\, \lambda e^{-\lambda x}\,dx$ [$\because$ R_1, R_2, R_3 are mutually independent]

$= \int_0^{\infty} \mathbb{P}[R_1 > 2x] \cdot \mathbb{P}[R_3 < \frac{2}{3}x] \cdot \lambda e^{-\lambda x}\,dx$ [$\because$ $R_1 \perp\!\!\!\perp R_3$]

$= \int_0^{\infty} e^{-\lambda \cdot 2x} \cdot (1 - e^{-\lambda \cdot \frac{2}{3}x}) \cdot \lambda e^{-\lambda x}\,dx$

$= \lambda \int_0^{\infty} e^{-3\lambda x}(1 - e^{-\frac{2\lambda x}{3}})\,dx$

$= \lambda \int_0^{\infty} e^{-3\lambda x} - \lambda \int_0^{\infty} e^{-\frac{11\lambda x}{3}}\,dx$

$= \lambda \cdot \frac{1}{3\lambda} - \lambda \cdot \frac{1}{\frac{11\lambda}{3}}$

$= \frac{1}{3} - \frac{3}{11}$

$= \frac{2}{33}$.

c) $R_1 > 2R_2 > \cdots > nR_n \iff \lambda R_1 > \lambda \cdot 2R_2 > \cdots > \lambda \cdot nR_n$ [$\because$ $\lambda > 0$]

[$\iff$ means if and only if]

So, $\mathbb{P}[R_1 > 2R_2 > \cdots > nR_n] = \mathbb{P}[\lambda R_1 > 2\lambda R_2 > \cdots > n\lambda R_n]$

Now, $\mathbb{P}[\lambda R_n \le x]$, for $x \ge 0$

$= \mathbb{P}[R_n \le \frac{x}{\lambda}]$

$= 1 - e^{-\lambda \cdot \frac{x}{\lambda}}$ [$\because$ $R_n \sim exp(\lambda)$]

$= 1 - e^{-x}$

$\mathbb{P}[\lambda R_n \le x]$, for $x < 0$

$= \mathbb{P}[R_n \le x/\lambda]$

$= 0$ [$\because$ $R_n \sim exp(\lambda)$]

$\therefore F_{\lambda R_n}(x) = \begin{cases} 1 - e^{-x}, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}$

$\Rightarrow \lambda R_n \sim exp(1)$, $\forall n \ge 3$.

Also, $R_1, \ldots, R_n$ are mutually independent $\Rightarrow \lambda R_1, \ldots, \lambda R_n$ are mutually independent

$\therefore \mathbb{P}[\lambda R_1 > \cdots > n\lambda R_n]$ is independent of λ, as the λR_n's are iid with exp(1)
which does not depend on λ.

Hence, $\mathbb{P}[R_1 > 2R_2 > \cdots > nR_n]$ does not depend on λ.

d) The R_k's being iid are exchangable.

So, $\mathbb{P}[R_1 < R_2 < \cdots < R_n]$ is same as the probability of any ordering of the rvs in increasing order, i.e.,

$$\mathbb{P}[R_1 < R_2 < \cdots < R_n] = \mathbb{P}[R_{k_1} < R_{k_2} < \cdots < R_{k_n}]$$

where the index set $\{k_1, \ldots, k_n\}$ is any reordering of $\{1, \ldots, n\}$.

All such reordering $\{R_{k_1} < \cdots < R_{k_n}\}$ and the events where tie occurs between two or more R_k's constitute the sample space Ω.

Since the R_k's are all continuous rvs, the probability of ties are zero

So, $$\sum_{\substack{\{k_1, \ldots, k_n\} \\ \text{is any ordering} \\ \text{of } \{1, \ldots, n\}}} \mathbb{P}[R_{k_1} < \cdots < R_{k_n}] = 1$$

$$\Rightarrow \mathbb{P}[R_{k_1} < \cdots < R_{k_n}] = \frac{1}{n!}$$

[$\because$ The no. of ways to reorder the set $\{1, .., n\}$ is $n!$]
and all such $\mathbb{P}[R_{k_1} < \cdots < R_{k_n}]$ has equal probability because the R_k's are iid]

Hence, $\mathbb{P}[R_1 < \cdots < R_n] = \frac{1}{n!}$

because $\{1, \ldots, n\}$ is one such reordering of $\{1, \ldots, n\}$.