
ENGINEERING PROBABILITY

HOMEWORK # 14:
Posted on 05/02/2018

Please work out the **ten** (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem **1.55** (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). **Show** work and **explain** reasoning.

1.

Let $U, U_1, \ldots, U_n$ be i.i.d. rvs $\Omega \to \mathbb{R}$, each of which is uniformly distributed over the unit interval $(0, 1)$.

1.a. Compute the probability $\mathbb{P}[U_1 \le U, \ldots, U_n \le U]$.

1.b. Are the rvs $\mathbf{1}[U_1 \le U], \ldots, \mathbf{1}[U_n \le U]$ mutually independent?

1.b. Evaluate the probabilities

$$\mathbb{P}\left[\sum_{\ell=1} \mathbf{1}[U_\ell \le U] = x\right], \quad x = 0, 1, \ldots, n$$

[HINT: Do these probabilities depend on x? Explain]

2.

Problem **4.44** (BT)

3.

It is known that the rv N is a conditionally Poisson rv given the rv Λ, i.e.,

$$\mathbb{P}[N = k | \Lambda = t] = \frac{t^k}{k!} e^{-t}, \qquad \begin{matrix} k = 0, 1, \ldots \\ t \ge 0 \end{matrix}$$

where the rv Λ is exponentially distributed with parameter $\lambda > 0$.

3.a. Compute the pmf of the rv N and identify this pmf.

3.b. For each $k = 0, 1, \ldots$, find the conditional pdf of Λ given $N = k$. Is this the pdf of a well-known distribution?

3.c. Evaluate the moment $\mathbb{E}[N\Lambda]$.

4.

The joint probability distribution of the two rvs X and Y admits the probability density function $f : \mathbb{R}^2 \to \mathbb{R}_+$ given by

$$f(x,y) = \begin{cases} x+y & \text{if } 0 < x,y < 1 \\ \\ 0 & \text{otherwise.} \end{cases}$$

4.a. Find the probability density functions $f_X, f_Y : \mathbb{R} \to \mathbb{R}_+$ of the rvs X and Y.

4.b. Are the rvs X and Y independent? Explain.

4.c. Evaluate the probability

$$\mathbb{P}\left[X+Y \leq t\right]$$

for t in the range $0 < t \leq 1$.

4.d. Compute $\mathbb{E}\left[X\right]$, $\mathrm{Var}[X]$ and $\mathrm{Cov}[X,Y]$.

5.

Let $\{X, X_k,\ k = 1,2,\ldots\}$ denote a collection of i.i.d. rvs defined on the same probability triple. Consider the rvs $\{Y_k,\ k = 1,2,\ldots\}$ defined by

$$Y_k = X_{k+1} - X_k, \quad k = 1,2,\ldots$$

For each $n = 1,2,\ldots$, write

$$a_n = \mathbb{P}\left[X_1 + \ldots + X_n > 0\right]$$

and

$$b_n = \mathbb{P}\left[Y_1 + \ldots + Y_n > 0\right].$$

5.a. Under the Gaussian assumption $X \sim \mathrm{N}(0,1)$, evaluate the probabilities a_n **(5 pts.)**, and b_n for each $n = 1,2,\ldots$.

5.b. With no distributional assumption on X other than $\mathbb{E}\left[X\right] = 0$ and $\mathbb{E}\left[|X|^2\right] = 1$, find the limit

$$\lim_{n\to\infty} \mathbb{P}\left[X_1 + \ldots + X_n > 0\right],$$

and use it to design an approximation to the probability

$$\mathbb{P}\left[X_1 + \ldots + X_n > 0\right]$$

when n is large. Explain your arguments carefully!

5.c. With no distributional assumption on X other than $\mathbb{E}\left[X\right] = 0$ and $\mathbb{E}\left[|X|^2\right] = 1$, find the limit

$$\lim_{n\to\infty} \mathbb{P}\left[Y_1 + \ldots + Y_n > 0\right].$$

Is this limiting value the same as the one obtained in Part **5.b**? Explain your arguments carefully!

6.

Throughout M and L are integers with $M \geq$ and $L \geq 2$. A box contains M distinct coins,[1] labelled $1, \ldots, M$, with coin m selected with probability a_m. Once a coin is selected and taken out of the box, it is tossed L times in succession under identical and independent conditions. If coin m had been selected, each toss would yield head (resp. tail) with probability p_m.

6.a. Construct a probability space to model this situation – Specify the sample space, the σ-field and the probability assignment explicitly.

6.b. With $\ell = 1, \ldots, L$, let E_ℓ denote the event that head is obtained in the ℓ^{th} toss of the selected coin. Are the events $E_1, \ldots, E_L$ mutually independent? Carefully explain your answer.

6.c. After the coin has been selected and taken out of the box, you watch the first coin toss. What is the posterior probability that coin m was selected if you observe a head after the first toss?

6.d. You walk away as the second coin toss is about to take place, but being the insatiably curious individual that you are, your mind wonders: What is the probability that this second coin toss also yields a head?

7.

The following setting occurs in the context of Queueing Theory: The rv N counts the number of arrivals to a service facility over some interval of time of *random* duration X. The theory stipulates that (i) the rv N is a geometric rv with parameter ρ, namely

$$\mathbb{P}[N = n] = (1-\rho)\rho^{n-1}, \quad n = 1, 2, \ldots$$

for some ρ in $(0, 1)$, and that (ii) for each $n = 1, 2, \ldots$, the conditional distribution of the rv X given $N = n$ admits a density given by

$$f_{X|N}(t|n) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}, \quad t \geq 0.$$

7.a. Compute the probability density function $f_X : \mathbb{R}_+ \to \mathbb{R}$ of the rv X. Is it a well known distribution?

7.b. For each $t > 0$, determine the conditional probability mass function of N given $X = t$, namely

$$\mathbb{P}[N = n | X = t], \quad n = 1, 2, \ldots$$

Is this conditional pmf related to a well-known pmf? If so, which one?

7.c. Use Parts **7.a** and **7.b** to compute $\mathbb{E}[XN]$ *explicitly.*

8.

Given are scalars a, b and c in $\mathbb{R}$ such that $a > 0$, $c > 0$ and $b^2 < ac$. It is determined that

[1] The coins may have different weights, may be made of different alloys, etc.

the rvs X and Y are jointly continuous with probability density function $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$f_{X,Y}(x, y) = \Gamma e^{-(ax^2 - 2bxy + cy^2)}, \quad x, y \in \mathbb{R}$$

for some $\Gamma > 0$.

8.a. Determine the value of Γ in terms of a, b and c.

8.b. Determine the probability distribution of the rv Y.

8.c. For every y in $\mathbb{R}$, determine the conditional probability distribution of X given $Y = y$.

8.d. Evaluate $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

8.e. Evaluate $\mathbb{E}[XY]$,

9.

The rv Λ is exponentially distributed with parameter $\lambda > 0$. You are being told that the discrete rv X is conditionally Poisson with parameter Λ, i.e., for each $t > 0$,

$$\mathbb{P}[X = x | \Lambda = t] = \frac{t^x}{x!} e^{-t}, \quad x = 0, 1, \ldots$$

9.a. Find the unconditional pmf of X, i.e., compute[2]

$$\mathbb{P}[X = x], \quad x = 0, 1, \ldots$$

Is this a well-known distribution?

9.b. With $x = 0, 1, \ldots$, find the conditional distribution of Λ given $X = x$.

9.c. Evaluate

$$\mathbb{E}\left[e^{-aX\Lambda} | X = x\right], \quad \begin{matrix} a > 0 \\ x = 0, 1, \ldots \end{matrix}$$

10.

Let X and Y be two independent rvs. Assume that the rvs X and Y are exponentially distributed with parameter $\lambda > 0$ and $\mu > 0$, respectively. A new rv R is now defined as

$$R = \begin{cases} \frac{\sqrt{X}}{\sqrt{X} + \sqrt{Y}} & \text{if } X > 0 \text{ and } Y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

10.a. Compute the cumulative distribution function of R, namely

$$\mathbb{P}[R \leq r], \quad r \in \mathbb{R}.$$

10.b. Is the rv R a continuous rv? If so, determine its probability density function $f_R : \mathbb{R} \to \mathbb{R}_+$.

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[2]**HINT:** Recall that $\int_0^\infty t^n e^{-t} dt = n!$ for each $n = 0, 1, \ldots$.

In what follows assume that we are in the symmetric case, namely $\lambda = \mu$.

10.c. Explain why $\mathbb{E}\left[R\right] = \frac{1}{2}$.

10.d. Use Part **10.c** to compute the covariance

$$\text{Cov}\left[R, \sqrt{X} + \sqrt{Y}\right].$$
