ENEE 324-01* / FALL 2018

HOMEWORK #14

SOLUTIONS

① (a) $\mathbb{P}[U_1 \le U, \ldots, U_n \le U]$

$= \mathbb{E}[\mathbb{1}[U_1 \le U, \ldots, U_n \le U]]$

$= \mathbb{E}(\mathbb{E}[\mathbb{1}(U_1 \le U, \ldots, U_n \le U] \mid U]]$ (Iterated conditioning)

$= \mathbb{E}(\mathbb{P}[U_1 \le U, \ldots, U_n \le U \mid U]]$

$= \mathbb{E}[\mathbb{P}[U_1 \le u, \ldots, U_n \le u \mid U = u]_{u=U}]$

$= \mathbb{E}[\mathbb{P}[U_1 \le u, \ldots, U_n \le u]_{u=U}]$ ($\because U \perp\!\!\!\perp U_i$, $i=1,\ldots,n$)

$= \mathbb{E}\left(\left[\prod_{i=1}^{n} \mathbb{P}(U_i \le u)\right]_{u=U}\right]$ ($U_i$'s are Mutually Independent)

$= \mathbb{E}[U^n \mathbb{1}[0 \le U < 1)]$

$= \int_0^1 u^n \, du = \left.\frac{u^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$ //

(b) For $1 \le i \le n$,

$\mathbb{P}[U_i \le U]$

$= \mathbb{E}[\mathbb{1}[U_i \le U)]$

$= \mathbb{E}(\mathbb{E}(\mathbb{1}(U_i \le U) \mid U]]$

$= \mathbb{E}(\mathbb{P}(U_i \le U \mid U)]$

$= \mathbb{E}[U]$

$= 1/2$

Hence, $\prod_{i=1}^{n} \mathbb{P}(U_i \le U) = \left(\frac{1}{2}\right)^n \neq \mathbb{P}[U_1 \le U, \ldots, U_n \le U]$

$\therefore$ The random variables $\mathbb{1}[U_1 \le U], \ldots, \mathbb{1}[U_n \le U]$ are not Mutually Independent.

(1.c) Consider $Y = \sum_{l=1}^{n} 1[U_l \le U]$. It is evident that the support of $Y$, $S_Y = \{0, 1, \ldots, n\}$ ($\because$ each indicator function can take only one of $\{0,1\}$ values)

In the event $Y = x$, out of the random variables $1[U_l \le U]$, $l = 1, \ldots, n$, $x$ are equal to one and the remaining $(n-x)$ rvs are zero. And since the random variables are iid, the probability of getting $x$ 1's and $(n-x)$ 0's is the same for all $\binom{n}{x}$ possible ways ~~to~~ of picking $x$.

$\therefore$ $\mathbb{P}[Y = x]$, $x \in \{0, 1, \ldots, n\}$

$$= \binom{n}{x} \mathbb{P}[U_1 \le U, U_2 \le U, \ldots, U_x \le U, U_{x+1} > U, \ldots, U_n > U]$$

$$= \binom{n}{x} \mathbb{E}\left[1[U_1 \le U, \ldots, U_x \le U, U_{x+1} > U, \ldots, U_n > U]\right]$$

$$= \binom{n}{x} \mathbb{E}\left[\mathbb{P}[U_1 \le U, \ldots, U_x \le U, U_{x+1} > U, \ldots, U_n > U \mid U]\right]$$

$$= \binom{n}{x} \mathbb{E}\left[\left(\prod_{i=1}^{x} \mathbb{P}[U_i \le u] \prod_{j=x+1}^{n} \mathbb{P}[U_j > u]\right)_{u=U}\right]$$

$$= \binom{n}{x} \mathbb{E}\left[U^x (1-U)^{n-x}\right]$$

$$= \binom{n}{x} \int_0^1 u^x (1-u)^{n-x} \, du$$

$$= \binom{n}{x} \left\{ \underbrace{\frac{-u^x (1-u)^{n-x+1}}{n-x+1} \Big|_0^1}_{=0} + \int_0^1 \frac{x}{n-x+1} u^{x-1} (1-u)^{n-x+1} \, du \right\}$$

$$= \binom{n}{x} \cdot \frac{x}{(n-x+1)} \int_0^1 u^{x-1} (1-u)^{n-x+1} \, du$$

$\vdots$ (By iterative integration by parts)

$$= \binom{n}{x} \times \frac{x!}{(n-x+1) \ldots n} \int_0^1 (1-u)^n \, du$$

$$= \frac{n!}{(n-x)!\, x!} \times \frac{x!\, (n-x)!}{n!} \times \frac{1}{n+1}$$

$$= \frac{1}{n+1} \quad //$$

This is a uniform distribution. The probabilities do not depend on $x$

③ (a) $g_\Lambda(t) = \lambda e^{-\lambda t}, \quad t > 0$

$$\therefore \mathbb{P}[N=k] = \int_0^\infty \mathbb{P}[N=k \mid \Lambda = t]\, g_\Lambda(t)\, dt, \quad k=0,1,\ldots$$

$$= \int_0^\infty \frac{t^k}{k!} e^{-t} \cdot \lambda e^{-\lambda t}\, dt$$

$$= \frac{\lambda}{k!} \int_0^\infty t^k e^{-(1+\lambda)t}\, dt$$

$$= \frac{\lambda}{k!(1+\lambda)} \int_0^\infty t^k (1+\lambda) e^{-(1+\lambda)t}\, dt$$

$$= \frac{\lambda}{k!(1+\lambda)} \left[ \frac{k!}{(1+\lambda)^k} \right] \quad \text{(By moments of the exponential distribution)}$$

$$= \frac{\lambda}{(1+\lambda)} \times \frac{1}{(1+\lambda)^k}$$

$$= \left(\frac{\lambda}{1+\lambda}\right) \left[ 1 - \left(\frac{\lambda}{1+\lambda}\right) \right]^k$$

$\Rightarrow$

$$N \sim \text{geometric}\left(\frac{\lambda}{1+\lambda}\right) \quad //$$

(b) $$f_{\Lambda|N}(t \mid N=k) = \frac{\mathbb{P}[N=k \mid \Lambda=t]\, g_\Lambda(t)}{\mathbb{P}[N=k]}, \quad t>0, \; k=0,1,2,\ldots$$

$$= \frac{\frac{t^k}{k!} e^{-t} \cdot \cancel{\lambda} e^{-\lambda t}}{\frac{\cancel{\lambda}}{(1+\lambda)^{k+1}}}$$

$$= \frac{t^k (1+\lambda)^{k+1} e^{-(1+\lambda)t}}{k!} \quad //$$

This is a Gamma distributions with parameters $\Gamma[k+1, 1+\lambda]$

(c) $\mathbb{E}(N\Lambda)$

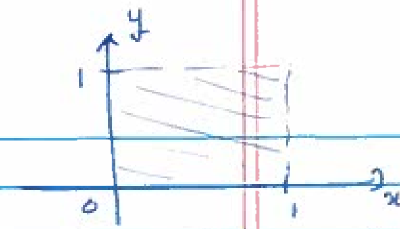
$= \mathbb{E}(\mathbb{E}[N\Lambda \mid \Lambda]]$ (Iterated conditioning)

$= \mathbb{E}[\Lambda \mathbb{E}[N \mid \Lambda]]$

$= \mathbb{E}[\Lambda^2]$ ($\because \mathbb{E}[N \mid \Lambda] = \Lambda$, $\because N$ is poission with parameter $\Lambda$)

$= \dfrac{2}{\lambda^2}$ //

④ $f(x,y) = \begin{cases} x+y & \text{if } 0 < x, y < 1 \\ 0 & o/\omega \end{cases}$




(4.a) $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^1 (x+y)\,dy \quad (\text{for } 0 < x < 1)$$

$$= \left(xy + y^2/2\right)_0^1$$

$$= x + 1/2$$

$$f_X(x) = \begin{cases} x + 1/2, & 0 < x < 1 \\ 0, & o/\omega \end{cases} \quad //$$

Similarly,

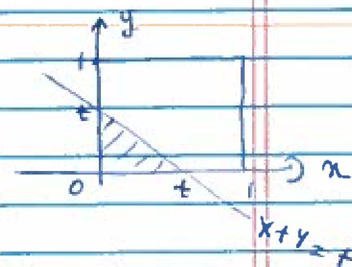
$$f_Y(y) = \begin{cases} y + 1/2, & 0 < y < 1 \\ 0, & o/\omega \end{cases} \quad //$$

(4.b) when $0 < x, y < 1$,

$$f_{XY}(x,y) = x + y \neq f_X(x)\, f_Y(y)$$

Hence, X and Y are not independent.

(4.c) $0 < t \leq 1$




$$\mathbb{P}[X + Y \leq t]$$

$$= \int_0^t \int_0^t (x+y)\,dy\,dx$$

$$= \int_0^t \left(xy + y^2/2\right)_0^t dx$$

$$= \int_0^t \left(xt + t^2/2\right) dx$$

$$= \left(\frac{t \cdot x^2}{2} + \frac{t^2}{2} \cdot x\right)_0^t$$

$$= t^3 \quad //$$

(4.d) $$\mathbb{E}(X) = \int_0^1 x f_X(x)\,dx$$

$$= \int_0^1 x(x + 1/2)\,dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1$$

$$= \frac{7}{12} \quad //$$

$$E(X^2) = \int_0^1 x^2\left(x + \tfrac{1}{2}\right) dx$$
$$= \int_0^1 \left(x^3 + x^2/_2\right) dx$$
$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1$$
$$= \frac{5}{12}$$

$$Var(X) = E(X^2) - (E(X))^2$$
$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2$$
$$= \frac{11}{144} \quad //$$

$$E(XY) = \int_0^1\int_0^1 xy\,(x+y)\,dx\,dy$$
$$= \int_0^1 y \int_0^1 \left(\frac{x^3}{3} + \frac{yx^2}{2}\right)_0^1 dy$$
$$= \int_0^1 y\left(\frac{1}{3} + \frac{y}{2}\right) dy$$
$$= \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$
$$= 1/3$$

$$Cov(XY) = E(XY) - E(X).E(Y)$$
$$= \frac{1}{3} - \left(\frac{7}{12}\right)^2$$
$$= \frac{48-49}{144}$$
$$= \frac{-1}{144} \quad //$$

(5) $\{X, X_k, k=1,2,\dots\}$ iid rvs and $Y_k = X_{k+1} - X_k$, $k=1,2,\dots$

(5.a) $X \sim N(0,1)$

Consider, $S_X = X_1 + \quad + X_n$. Since $X_i \sim N(0,1)$, iid

$$S_X \sim N(0, n)$$

$$\therefore a_n = \mathbb{P}[S_X > 0]$$
$$= 1 - \Phi\left(\frac{0}{n}\right)$$
$$= 1 - \Phi(0)$$
$$= 1/2$$

$$S_Y = Y_1 + \dots + Y_n$$
$$= (X_2 - X_1) + (X_3 - X_2) + \dots + (X_{n+1} - X_n)$$
$$= X_{n+1} - X_1$$

$$\Rightarrow S_Y \sim N(0, 2) \quad \text{for } n=1,2,\dots$$

Hence, $b_n = \mathbb{P}[S_Y > 0]$
$$= 1 - \Phi(0)$$
$$= 1/2$$

(5.b) $\mathbb{E}(X) = 0$, $\mathbb{E}[|X|^2] = 1$

The Central Limit Theorem: If $\mathbb{E}(X) = \mu$, $Var(X) = \sigma^2$,

$$Z_n = \frac{X_1 + \dots + X_n - n\mu}{\sigma\sqrt{n}}$$

Then,

$$\lim_{n\to\infty} \mathbb{P}[Z_n \le z] = \Phi(z)$$ (The CDF of $Z_n$ converges to the standard normal CDF)

In the given problem, $\mathbb{E}(X) = 0$ and $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1$

$$\Rightarrow Z_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}$$

By the Central Limit Theorem, $\lim_{n\to\infty} \mathbb{P}\left[\frac{X_1 + \dots + X_n}{\sqrt{n}} \le z\right] = \Phi(z)$

In particularly when $z = 0$;

$$\lim_{n \to \infty} \mathbb{P}\left[\frac{X_1 + \dots + X_n}{\sqrt{n}} \leq 0\right] = \Phi(0)$$

$$\lim_{n \to \infty} \mathbb{P}\left[X_1 + \dots + X_n \leq 0\right] = \Phi(0) = 1/2$$

$$\Rightarrow \lim_{n \to \infty} \mathbb{P}\left[X_1 + \dots + X_n > 0\right] = 1/2 \;//$$

Hence, when $n$ is large;

$$\mathbb{P}\left[X_1 + \dots + X_n > 0\right] \simeq 1/2 \;//$$

(5.c) $S_Y = Y_1 + \dots + Y_n$
$= X_{n+1} - X_1$

$\mathbb{P}[S_Y > 0]$
$= \mathbb{P}[X_{n+1} - X_1 > 0]$
$= \mathbb{P}[X_{n+1} > X_1]$

Since $X_1, \dots, X_n, \dots$ are iid; $\mathbb{P}(X_{n+1} > X_1) = \mathbb{P}(X_1 > X_{n+1})$

But,

$$\mathbb{P}(X_{n+1} > X_1) + \mathbb{P}(X_{n+1} < X_1) + \underbrace{\mathbb{P}(X_{n+1} = X_1)}_{\to 0} = 1$$

$\Rightarrow$ (Assuming $X$ is continous)

$$\mathbb{P}(X_{n+1} > X_1) = \mathbb{P}(X_{n+1} < X_1) = 1/2$$

$$\therefore \mathbb{P}[Y_1 + \dots + Y_n > 0] = 1/2 \quad \text{for } \forall n$$

Hence

$$\lim_{n \to \infty} \mathbb{P}[Y_1 + \dots + Y_n > 0] = 1/2 \;//$$

This value is the same as the one obtained in 5.b.

Note that in this case the claim is true for all $n$, not only in the limiting value!

(6) The sample space can be constructed as the ordered pairs (Choice of coin, Sequence of observations)

(a)

$$\therefore \Omega = \left\{ \left[ m, (y_1, y_2, \ldots, y_L) \right], \ 1 \leq m \leq M, \ \underset{1 \leq i \leq L}{y_i \in \{0,1\}} \right\}$$

(We represent an observation of Heads by 1 and tails by 0).

If the event of selecting coin $m$ is $A_m$,

$$\mathbb{P}\left[ (y_1, y_2, \ldots, y_L) \mid A_m \right] = p_m^{\sum_{i=1}^{L} y_i} (1 - p_m)^{L - \sum_{i=1}^{L} y_i} \quad \text{and}$$

$$\mathbb{P}\left[ A_m \right] = a_m$$

Hence, the probability of the singleton events in the Sample space;

$$\begin{aligned} \mathbb{P}[\{\omega\}] &= \mathbb{P}\left[ \left[ m, (y_1, y_2, \ldots, y_L) \right] \right] \\ &= \mathbb{P}\left[ (y_1, y_2, \ldots, y_L) \mid A_m \right] \mathbb{P}(A_m] \\ &= a_m \cdot p_m^{\sum_{i=1}^{L} y_i} (1 - p_m)^{L - \sum_{i=1}^{L} y_i} \quad // \end{aligned}$$

Since $\Omega$ is countable, $\mathcal{F}$ = Power Set $(2^{\Omega})$

(b)
$$\begin{aligned} \mathbb{P}[E_\ell] &= \sum_{m=1}^{M} \mathbb{P}[E_\ell \mid A_m] \, \mathbb{P}(A_m) \quad \text{(Law of Total Probabilities)} \\ &= \sum_{m=1}^{M} p_m \cdot a_m \quad \text{for } 1 \leq \ell \leq L \end{aligned}$$

$$\begin{aligned} \mathbb{P}[E_1, E_2, \ldots, E_L] &= \sum_{m=1}^{M} \mathbb{P}\left[ E_1, E_2, \ldots, E_L \mid A_m \right] \mathbb{P}\left[ A_m \right] \\ &= \sum_{m=1}^{M} p_m^{L} \cdot a_m \\ &\neq \left( \sum_{m=1}^{M} p_m a_m \right)^{L} = \prod_{\ell=1}^{L} \mathbb{P}(E_\ell) \end{aligned}$$

Hence, the events $E_1, E_2, \ldots, E_L$ are not mutually independent.

(c) $\mathbb{P}[A_m \mid E_1] = \dfrac{\mathbb{P}[E_1 / A_m]\ \mathbb{P}[A_m]}{\mathbb{P}[E_1]}$

$= \dfrac{p_m a_m}{\sum_{k=1}^{M} p_k a_k}$ (from (b))

//

(d) $\mathbb{P}[E_2 \mid E_1] = \dfrac{\mathbb{P}(E_1, E_2)}{\mathbb{P}(E_2)}$

$= \dfrac{\sum_{m=1}^{M} p_m^2 a_m}{\sum_{m=1}^{M} p_m a_m}$ (from (b))

//

7) $\mathbb{P}[N=n] = (1-p)p^{n-1}, \quad n=1,2,\dots$

$$f_{X|N}(t|n) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}, \quad t \geq 0, \quad n=1,2,\dots$$

a) We can write pdf of N as:

$$f_N(n) = \sum_{n_0=1}^{\infty} \mathbb{P}[N=n] \cdot \delta(n-n_0) \qquad \text{[where '}\delta\text{' is the Dirrac-delta f}^n\text{]}$$

Now, for $t \geq 0$,

$$f_X(t) = \int_{n=-\infty}^{\infty} f_{X,N}(t,n)\, dn$$

$$= \int_{n=-\infty}^{\infty} f_{X|N}(t|n) \cdot f_N(n)\, dn$$

$$= \int_{n=-\infty}^{\infty} \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} \cdot \sum_{n_0=1}^{\infty} \mathbb{P}[N=n] \cdot \delta(n-n_0)\, dn$$

$$= \int_{n=-\infty}^{\infty} \sum_{n_0=1}^{\infty} \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} (1-p) p^{n-1} \delta(n-n_0)\, dn$$

$$= \sum_{n_0=1}^{\infty} \int_{n=-\infty}^{\infty} \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} (1-p) p^{n-1} \delta(n-n_0)\, dn$$ [by Fubini's theorem, we can interchange the sum and integral if $|f_X(t)| < \infty$]

$$= \sum_{n_0=1}^{\infty} \lambda \frac{(\lambda t)^{n_0-1}}{(n_0-1)!} e^{-\lambda t} (1-p) p^{n_0-1}$$ [by property of Delta f$^n$]

$$= \lambda e^{-\lambda t}(1-p) \sum_{n_0=1}^{\infty} \frac{(\lambda t p)^{n_0-1}}{(n_0-1)!}$$

$$= \lambda e^{-\lambda t}(1-p) \sum_{m=0}^{\infty} \frac{(\lambda t p)^m}{m!}$$ [changing the index using $m = n_0 - 1$]

$$= \lambda e^{-\lambda t}(1-p) \cdot e^{\lambda t p}$$

$$= \lambda(1-p) \cdot e^{-\lambda(1-p)t}$$

Clearly, $f_X(t) = 0$ for $t < 0$ because $f_{X|N}(t|n) = 0$ if $t < 0$ and $f_X(t) = \int_{n=-\infty}^{\infty} f_{X|N}(t|n) f_N(n)\, dn$.

So, $X \sim \exp(\lambda(1-p))$.

b) For each $t>0$, $n=1,2,\ldots$,

$$\mathbb{P}[N=n \mid X=t] = \frac{f_{X|N}(t|n)\cdot \mathbb{P}[N=n]}{f_X(t)}$$

$$= \frac{\lambda(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}\cdot (1-p)p^{n-1}\cdot \frac{1}{\lambda(1-p)e^{-\lambda t(1-p)}}$$

$$= \frac{(\lambda t p)^{n-1}}{(n-1)!} e^{-\lambda t p}, \quad n=1,2,\ldots$$

So, $\mathbb{P}[N=1+m \mid X=t] = \frac{(\lambda t p)^m}{m!} e^{-\lambda t p}$, $m=0,1,2,\ldots$

$$\Rightarrow \mathbb{P}[N-1=m \mid X=t] = \frac{(\lambda t p)^m}{m!} e^{-\lambda t p}, \quad m=0,1,2,\ldots$$

Thus, the conditional pmf of $N$ given $X=t$, $t>0$, is related to Poisson distribution.
In particular,
the rv $(N-1) \mid X=t \sim \text{Pois}(\lambda t p)$, $t>0$

c) $$\mathbb{E}[XN] = \int_{t=-\infty}^{\infty} \sum_{n=1}^{\infty} tn\cdot \mathbb{P}[N=n \mid X=t]\cdot f_X(t)\,dt$$ [by def$^{n}$ of expectation of product of two rvs]

$$= \int_{t=0}^{\infty} \sum_{n=1}^{\infty} tn\cdot \frac{(\lambda t p)^{n-1}}{(n-1)!} e^{-\lambda t p}\cdot \lambda(1-p)e^{-\lambda(1-p)t}\,dt$$ [by results obtained in (7.a) & (7.b)]

$$= \int_{t=0}^{\infty} \sum_{n=1}^{\infty} tn \frac{(\lambda t p)^{n-1}}{(n-1)!} \lambda(1-p)e^{-\lambda t}\,dt$$

$$= \int_{t=0}^{\infty} \left[\sum_{n=1}^{\infty} n \frac{(\lambda t p)^{n-1}}{(n-1)!}\right]\cdot t\lambda(1-p)e^{-\lambda t}\,dt$$

Now, $$\sum_{n=1}^{\infty} n\cdot \frac{(\lambda t p)^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} n\cdot \frac{\mu^{n-1}}{(n-1)!}$$ [where $\mu := \lambda t p$]

$$= \sum_{m=0}^{\infty} (m+1)\cdot \frac{\mu^m}{m!}$$ [where $m := n-1$]

$$= \sum_{m=0}^{\infty} m\frac{\mu^m}{m!} + \sum_{m=0}^{\infty} \frac{\mu^m}{m!}$$

$$= \sum_{m=1}^{\infty} m\frac{\mu^m}{m!} + e^{\mu}$$

$$= \sum_{m=1}^{\infty} \frac{\mu^m}{(m-1)!} + e^{\mu}$$

$$= \mu \sum_{m=1}^{\infty} \frac{\mu^{m-1}}{(m-1)!} + e^{\mu}$$

$$= \mu e^{\mu} + e^{\mu}$$

$$= \lambda t p . e^{\lambda t p} + e^{\lambda t p}$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

By differentiating both sides w.r.t. $x$

$$\Rightarrow e^x = \sum_{m=0}^{\infty} \frac{m x^{m-1}}{m!}$$

$$= \sum_{m=1}^{\infty} \frac{m x^{m-1}}{m!}$$

$$= \sum_{m=1}^{\infty} \frac{x^{m-1}}{(m-1)!}$$

So, $\mathbb{E}[XN] = \int_{t=0}^{\infty} e^{\lambda t p}(\lambda t p+1) . t\lambda(1-p)e^{-\lambda t}\, dt$

$$= \lambda p \int_0^{\infty} t^2 \lambda(1-p) e^{-\lambda(1-p)t}\, dt + \int_0^{\infty} t . \lambda(1-p) e^{-\lambda(1-p)t}\, dt$$

$$= \lambda p . \mathbb{E}[X^2] + \mathbb{E}[X]$$

$$= \lambda p . \frac{2}{[\lambda(1-p)]^2} + \frac{1}{\lambda(1-p)}$$

[$\because$ The first integrand is $2^{nd}$ moment of exponential distribution with parameter $\lambda(1-p)$ and the second integrand is mean of $\sim exp(\lambda(1-p))$]

$$= p . \frac{2}{\lambda(1-p)^2} + \frac{1}{\lambda(1-p)}$$

$$= \frac{1}{\lambda(1-p)}\left(\frac{2p}{1-p} + 1\right)$$

$$= \frac{(1+p)}{(1-p)^2} \frac{1}{\lambda}.$$

8) $f_{X,Y}(x,y) = \Gamma . e^{-(ax^2+cy^2-2bxy)}$, $\forall\, x, y \in \mathbb{R}$, $\Gamma > 0$.

$a, c > 0$, $b^2 < ac$.

a) $\int_x\int_y f_{X,Y}(x,y)\,dx\,dy = 1 \Rightarrow \int_x\int_y \Gamma . e^{-(ax^2+cy^2-2bxy)}\,dx\,dy = 1$

$$\Rightarrow \frac{1}{\Gamma} = \int_x\int_y e^{-(ax^2+cy^2-2bxy)}\,dx\,dy$$

Now, by property of Bivariate Normal distribution,

$$1 = \int_x\int_y \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-p^2}} e^{-\frac{1}{2(1-p^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2p\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right\}}\,dx\,dy$$

$$\Rightarrow \int_x\int_y e^{-\frac{1}{2(1-p^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2p\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right\}}\,dx\,dy = 2\pi\sigma_x\sigma_y\sqrt{1-p^2}$$

where $\mu_x, \mu_y \in \mathbb{R}$, $\sigma_x, \sigma_y > 0$, $p^2 < 1$.

Comparing, $ax^2+by^2-2bxy = \left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right\}\cdot\frac{1}{2(1-\rho^2)}$

we get,

$$\mu_x=\mu_y=0$$

$$a=\frac{1}{\sigma_x^2\cdot 2(1-\rho^2)}$$

$$c=\frac{1}{\sigma_y^2\cdot 2(1-\rho^2)}$$

$$b=\frac{\rho}{\sigma_x\sigma_y\cdot 2(1-\rho^2)}$$

So, $ac=\frac{1}{\sigma_x^2\sigma_y^2(2(1-\rho^2))^2}$

$$\Rightarrow \sqrt{ac}=\pm\frac{1}{\sigma_x\sigma_y\cdot 2(1-\rho^2)}$$

$$\Rightarrow b=\pm\rho\sqrt{ac}$$

$$\Rightarrow \rho=\pm\frac{b}{\sqrt{ac}}$$

$$\Rightarrow \rho^2=\frac{b^2}{ac}<1 \quad [\because b^2<ac]$$

Hence, $\int_x\int_y e^{-(ax^2+by^2-2bxy)}\,dx\,dy = 2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}$

Now, $b=\frac{\rho}{\sigma_x\sigma_y\cdot 2(1-\rho^2)} \Rightarrow \sigma_x\sigma_y\cdot 2\pi\sqrt{1-\rho^2}=\frac{\rho\pi}{b\sqrt{1-\rho^2}}$

$$=\frac{b}{\sqrt{ac}}\pi\cdot\frac{1}{b\sqrt{1-\frac{b^2}{ac}}}$$

$$=\frac{\pi}{\sqrt{ac-b^2}}$$

$$\therefore \int_x\int_y e^{-(ax^2+by^2-2bxy)}\,dx\,dy=\frac{\pi}{\sqrt{ac-b^2}}$$

$$\Rightarrow \frac{1}{\Gamma}=\frac{\pi}{\sqrt{ac-b^2}}$$

$$\Rightarrow \Gamma=\frac{\sqrt{ac-b^2}}{\pi}.$$

$$\therefore f_{X,Y}(x,y)=\frac{\sqrt{ac-b^2}}{\pi}e^{-(ax^2+cy^2-2bxy)},\ \forall x,y\in\mathbb{R}.$$

b) $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \qquad , y \in \mathbb{R}.$

$$= \frac{\sqrt{ac-b^2}}{\pi}\int_{-\infty}^{\infty} e^{-(ax^2+cy^2-2bxy)}\,dx$$

$$= \frac{\sqrt{ac-b^2}}{\pi}\cdot e^{-cy^2}\int_{-\infty}^{\infty} e^{-(ax^2-x.2by)}\,dx$$

Now, $(ax^2-x.2by) = \left(x\sqrt{a}-\frac{yb}{\sqrt{a}}\right)^2-\left(\frac{yb}{\sqrt{a}}\right)^2$

$$\therefore f_Y(y) = \frac{\sqrt{ac-b^2}}{\pi}e^{-cy^2}.e^{\left(\frac{yb}{\sqrt{a}}\right)^2}\int_{-\infty}^{\infty} e^{-\left(x\sqrt{a}-\frac{yb}{\sqrt{a}}\right)^2}dx$$

$$= \frac{\sqrt{ac-b^2}}{\pi}e^{-y^2\left(c-\frac{b^2}{a}\right)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\frac{1}{\sqrt{2a}}}-\frac{\frac{by}{a}}{\frac{1}{\sqrt{2a}}}\right)^2}dx$$

$$= \frac{\sqrt{ac-b^2}}{\pi}e^{-y^2\left(c-\frac{b^2}{a}\right)}.\sqrt{2\pi}.\frac{1}{\sqrt{2a}} \qquad \left[\because 1 = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx,\ \text{take } \sigma=\frac{1}{\sqrt{2a}},\ \mu=\frac{by}{a}\right]$$

$$= \sqrt{\frac{ac-b^2}{\pi a}}\,e^{-y^2\left(c-\frac{b^2}{a}\right)} \quad ,\forall y\in\mathbb{R}.$$

c) For each $y \in \mathbb{R}$,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \quad ,\forall x\in\mathbb{R}$$

$$= \frac{\frac{\sqrt{ac-b^2}}{\pi}e^{-(ax^2+cy^2-2bxy)}}{\sqrt{\frac{ac-b^2}{\pi a}}\,e^{-y^2\left(c-\frac{b^2}{a}\right)}}$$

$$= \sqrt{\frac{a}{\pi}}\,e^{-\left(ax^2-2bxy-\frac{b^2y^2}{a}\right)}, \quad x\in\mathbb{R}.$$

d) $\mathbb{E}[Y] = \int_{-\infty}^{\infty} y\,f_Y(y)\,dy = \int_{-\infty}^{\infty} y.\sqrt{\frac{ac-b^2}{\pi a}}\,e^{-y^2\left(c-\frac{b^2}{a}\right)}dy$

$= 0.$ [since the integrand is an odd $f^n$ w.r.t. $y$]

By symmetry of the problem in $x$ and $y$,

$$f_X(x) = \sqrt{\frac{ac-b^2}{\pi c}}\,e^{-x^2(a-b^2/c)}, \quad \forall x\in\mathbb{R}$$

and $\mathbb{E}[X]=0$ by a similar argument as $\mathbb{E}[Y]$.

e) $\mathbb{E}[XY] = \int_x\int_y xy\, f_{x,y}(x,y)\,dx\,dy$ [Be careful here as to not write $\int_x\int_y xy f_x(x) f_y(y)\,dx\,dy$]

$= \iint xy\cdot\Gamma\cdot e^{-(ax^2-2bxy+cy^2)}\,dx\,dy$

$= \Gamma\int_{y=-\infty}^{\infty} y e^{-cy^2}\left[\int_{x=-\infty}^{\infty} xe^{-(ax^2-2bxy)}dx\right]dy$

As shown in the part (8·b), the term inside the [ ] bracket, can be calculated by:

$$\frac{1}{\sqrt{2\pi}\cdot\frac{1}{\sqrt{2a}}}\int_{-\infty}^{\infty} xe^{-(ax^2-2bxy)}dx = \frac{1}{\sqrt{2\pi}\cdot\frac{1}{\sqrt{2a}}}e^{\frac{y^2b^2}{a}}\int_{-\infty}^{\infty} xe^{-\frac{1}{2}\left(\frac{x}{\frac{1}{\sqrt{2a}}}-\frac{yb/a}{\frac{1}{\sqrt{2a}}}\right)^2}dx$$

$= e^{\frac{y^2b^2}{a}}\cdot\frac{yb}{a}$ [by expectation of a Normal rv. with $\sigma=\frac{1}{\sqrt{2a}}, \mu=\frac{y}{a}$

$$\Rightarrow \int_{-\infty}^{\infty} xe^{-(ax^2-2bxy)}dx = \sqrt{\frac{a}{\pi}}\cdot e^{\frac{y^2b^2}{a}}\cdot\frac{yb}{a}$$

$$\therefore \mathbb{E}[XY] = \Gamma\int_{-\infty}^{\infty} ye^{-cy^2}\cdot\sqrt{\frac{a}{\pi}}\,e^{\frac{y^2b^2}{a}}\cdot\frac{yb}{a}\,dy$$

$$= \Gamma\cdot\frac{b}{a}\cdot\sqrt{\frac{a}{\pi}}\int_{-\infty}^{\infty} y^2e^{-y^2\left(c-\frac{b^2}{a}\right)}dy$$

$$= \Gamma\cdot\frac{b}{\sqrt{a\pi}}\int_{-\infty}^{\infty} y^2\cdot e^{-\frac{1}{2}\cdot y^2\cdot 2\left(c-\frac{b^2}{a}\right)}dy$$

$$= \Gamma\cdot\frac{b}{\sqrt{a\pi}}\cdot\sqrt{2\pi}\cdot\sigma\int_{-\infty}^{\infty} y^2\cdot\frac{1}{\sqrt{2\pi}\cdot\sigma}e^{-\frac{y^2}{2\sigma^2}}dy \quad \left[\text{where } \sigma^2=\frac{1}{2\left(c-\frac{b^2}{a}\right)}\right]$$

$$= \Gamma\cdot\frac{b}{\sqrt{a\pi}}\cdot\sqrt{2\pi}\cdot\sigma\cdot(\sigma^2+0)$$ [$\because$ The integral is $2^{nd}$ moment of a Normal rv with mean 0 and variance $\sigma^2$]

$$= \Gamma\cdot\frac{b}{\sqrt{a}}\cdot\sqrt{2}\,\sigma^3 = \frac{ab}{2\pi(ac-b^2)}.$$

9) An useful result:

$$\int_0^\infty e^{-\alpha t}.t^{(n-1)}dt = \frac{(n-1)!}{\alpha^n}, \text{ where } \alpha>0,\ n=1,\ldots \qquad \text{——— (A)}$$

Proof:

Let $y=\alpha t$

Then $\int_0^\infty e^{-\alpha t}.t^{(n-1)}dt = \int_0^\infty e^{-y}y^{n-1}.\alpha^{-(n-1)}.\alpha^{-1}dy$

$$= \alpha^{-n}\int_0^\infty e^{-y}.y^{n-1}dy$$

$$= \alpha^{-n}.(n-1)! \quad \left[\text{Recall: } \int_0^\infty t^n e^{-t}dt = n!,\ n=0,1,\ldots\right]$$

Now the problem:

$$\Lambda \sim \exp(\lambda), \lambda>0$$

$$\Rightarrow f_\Lambda(t) = \begin{cases} \lambda e^{-\lambda t}, & t\geq 0 \\ 0, & \text{o/w} \end{cases}$$

For each $t>0$,

$$\mathbb{P}[X=x \mid \Lambda=t] = \frac{t^x}{x!}e^{-t}, \quad x=0,1,\ldots$$

a) $\mathbb{P}[X=x]$, $x=0,1,\ldots$

$$= \int_{-\infty}^\infty \mathbb{P}[X=x \mid \Lambda=t]f_\Lambda(t)dt$$

$$= \int_0^\infty \frac{t^x}{x!}e^{-t}.\lambda e^{-\lambda t}dt$$

$$= \frac{\lambda}{x!}\int_0^\infty t^x.e^{-(\lambda+1)t}dt$$

$$= \frac{\lambda}{x!}.\frac{x!}{(\lambda+1)^{x+1}} \quad [\text{using (A) with } n-1=x,\ \alpha=\lambda+1>0]$$

$$= \frac{\lambda}{(\lambda+1)^{x+1}}$$

$$= \frac{\lambda}{\lambda+1}.\left(\frac{1}{\lambda+1}\right)^x$$

$$= \frac{\lambda}{\lambda+1}\left(1-\frac{\lambda}{\lambda+1}\right)^x, \quad x=0,1,\ldots$$

$\therefore X \sim \text{Geo}\left(\frac{\lambda}{\lambda+1}\right)$ on $\mathbb{N}_0$.

b) With $x = 0, 1, \ldots,$

$$f_{\Lambda|X}(t|x) = \frac{\mathbb{P}[X=x|\Lambda=t]\, f_\Lambda(t)}{\mathbb{P}[X=x]}$$

$\therefore f_{\Lambda|X}(t|x) = 0$ if $t<0$ $[\because f_\Lambda(t) = 0$ if $t<0]$

For $t \geq 0$,

$$f_{\Lambda|X}(t|x) = \frac{\frac{t^x}{x!}e^{-t}\cdot \lambda e^{-\lambda t}}{\frac{\lambda}{(\lambda+1)^{x+1}}}$$

$$= \frac{t^x}{x!}e^{-(\lambda+1)t}\cdot(\lambda+1)^{x+1}$$

$$= \frac{[(\lambda+1)t]^x}{x!}e^{-(\lambda+1)t}\cdot(\lambda+1).$$

c) $\mathbb{E}[e^{-aX\Lambda} | X=x]$, $x = 0, 1, \ldots; a > 0$

$$= \int_{t=-\infty}^{\infty} e^{-axt}\cdot f_{\Lambda|X}(t|x)\,dt$$

$$= \int_{t=0}^{\infty} e^{-axt}\cdot\frac{t^x}{x!}e^{-(\lambda+1)t}\cdot(\lambda+1)^{x+1}\,dt \quad [\text{from result of } (9.b)]$$

$$= \frac{(\lambda+1)^{x+1}}{x!}\int_0^\infty e^{-(ax+\lambda+1)t}\cdot t^x\,dt$$

$$= \frac{(\lambda+1)^{x+1}}{x!}\cdot\frac{x!}{(ax+\lambda+1)^{x+1}} \quad [\text{from } (A)]$$

$$= \left(\frac{\lambda+1}{ax+\lambda+1}\right)^{x+1}, \quad x = 0, 1, \ldots$$

10) $X \sim \exp(\lambda)$

$Y \sim \exp(\mu)$

$X \perp\!\!\!\perp Y$

$\lambda, \mu > 0$

$$R = \begin{cases} \frac{\sqrt{X}}{\sqrt{X}+\sqrt{Y}} & \text{, if } X>0, Y>0 \\ 0 & \text{, o/w} \end{cases}$$

a) $X \sim \exp(\lambda) \Rightarrow \mathbb{P}[X>0] = 1$

$Y \sim \exp(\mu) \Rightarrow \mathbb{P}[Y>0] = 1$

$X \perp\!\!\!\perp Y \Rightarrow \mathbb{P}[X>0, Y>0] = \mathbb{P}[X>0].\mathbb{P}(Y>0] = 1$

$\Rightarrow \mathbb{P}[\{X>0, Y>0\}^c] = 0.$

Clearly,

$\mathbb{P}[R \leq r] = 0$, if $r \leq 0$

and $\mathbb{P}[R \leq r] = 1$, if $r \geq 1$

For $r \in (0,1)$,

$$\mathbb{P}[R \leq r] = \mathbb{P}[R \leq r, \{X>0, Y>0\}] + \mathbb{P}[R \leq r, \{X>0, Y>0\}^c]$$

$$= \mathbb{P}[R \leq r, \{X>0, Y>0\}] + 0 \quad [\because R \leq r, \{X>0, Y>0\}^c \subseteq \{X>0, Y>0\}^c \text{ and } \mathbb{P}[\{X>0, Y>0\}^c] = 0]$$

$$= \mathbb{P}\left[\frac{\sqrt{X}}{\sqrt{X}+\sqrt{Y}} \leq r, X>0, Y>0\right]$$

$$= \mathbb{P}[\sqrt{X}(1-r) \leq r\sqrt{Y}, X>0, Y>0]$$

$$= \mathbb{P}[X(1-r)^2 \leq r^2 Y, X>0, Y>0]$$

$$= \mathbb{P}[X \leq (\tfrac{r}{1-r})^2 Y, X>0, Y>0]$$

$$= \mathbb{P}[Y>0, 0 < X \leq (\tfrac{r}{1-r})^2 Y]$$

$$= \mathbb{E}[\mathbb{1}[Y>0, 0<X<(\tfrac{r}{1-r})^2 Y]]$$

$$= \mathbb{E}\left[\left(\mathbb{E}[\mathbb{1}[Y>0].\mathbb{1}[0<X \leq (\tfrac{r}{1-r})^2 Y] \mid Y=y]\right)_{y=Y}\right]$$

$$= \mathbb{E}\left[\left(\mathbb{E}[\mathbb{1}[y>0].\mathbb{1}[0<X \leq (\tfrac{r}{1-r})^2 y] \mid Y=y]\right)_{y=Y}\right]$$

$$= \mathbb{E}\left[\left(\mathbb{1}[y>0]\cdot \mathbb{E}\left[\mathbb{1}\left[0<X\leq \left(\frac{r}{1-r}\right)^2 y\right]\right)_{y=Y}\right] \quad [\because X \perp Y]$$

$$= \mathbb{E}\left[\mathbb{1}[Y>0]\left(\mathbb{P}\left[0<X\leq \left(\frac{r}{1-r}\right)^2 y\right)_{y=Y}\right]$$

$$= \mathbb{E}\left[\mathbb{1}[Y>0].\left(\mathbb{P}\left[X\leq \left(\frac{r}{1-r}\right)^2 y\right]\right)_{y=Y}\right] \quad [\because X\sim \exp(\lambda) \Rightarrow \mathbb{P}[X\leq x]=0 \ \forall x<0]$$

$$= \mathbb{E}\left[\mathbb{1}[Y>0]\left(1-e^{-\lambda\left(\frac{r}{1-r}\right)^2 y}\right)_{y=Y}\right] \quad [\because X\sim \exp(\lambda) \Rightarrow \mathbb{P}[X\leq x]=1-e^{-\lambda x},\ x>0]$$

$$= \mathbb{E}\left[\mathbb{1}[Y>0].\left(1-e^{-\lambda\left(\frac{r}{1-r}\right)^2 Y}\right]\right.$$

$$= \int_0^\infty \left(1-e^{-\lambda\left(\frac{r}{1-r}\right)^2 y}\right)\mu e^{-\mu y}\,dy$$

$$= \int_0^\infty \mu e^{-\mu y}\,dy - \int_0^\infty \mu e^{-\left(\lambda\left(\frac{r}{1-r}\right)^2+\mu\right)y}\,dy$$

$$= 1-\frac{\mu}{\mu+\lambda\left(\frac{r}{1-r}\right)^2}$$

So, $\mathbb{P}[R\leq r] = \begin{cases} 0, & \text{if } r\leq 0 \\ 1-\frac{\mu}{\mu+\lambda\left(\frac{r}{1-r}\right)^2}, & \text{if } 0<r<1 \\ 1, & \text{if } r\geq 1\end{cases}$

b) Clearly $F_R(r)=\mathbb{P}[R\leq r]$ is continuous and differentiable w.r.t. $r$.

So, $R$ is a continuous r.v.

$$f_R(r) = \frac{d}{dr}F_R(r) = \frac{d}{dr}\left(1-\frac{\mu}{\mu+\lambda\left(\frac{r}{1-r}\right)^2}\right), \quad 0<r<1$$

$$= \frac{d}{dr}\left(\frac{\lambda r^2}{r^2\lambda+\mu(1-r)^2}\right), \quad 0<r<1$$

$$= \begin{cases} 2\mu\lambda\frac{r(1-r)}{(\lambda r^2+\mu(1-r)^2)^2}, & 0<r<1 \\ 0, & \text{o/w.}\end{cases}$$

Now, $\lambda = \mu$.

c) $$F_R(r) = \begin{cases} 0 & , \quad r \le 0 \\ 1 - \dfrac{1}{1+\left(\frac{r}{1-r}\right)^2} & , \quad 0 < r < 1 \\ 1 & , \quad r \ge 1 \end{cases}$$

If $|x| > 1/2$,

$$\mathbb{P}\left[R \le \tfrac{1}{2} - |x|\right] = 0 \qquad \left[\because \ \tfrac{1}{2} - |x| < 0\right]$$

and $$\mathbb{P}\left[R \ge \tfrac{1}{2} + |x|\right] = 1 - \mathbb{P}\left[R < \tfrac{1}{2} + |x|\right]$$
$$= 1 - 1 \qquad \left[\because \ \tfrac{1}{2} + |x| > 1\right]$$
$$= 0.$$

$$\Rightarrow \mathbb{P}\left[R \le \tfrac{1}{2} - |x|\right] = \mathbb{P}\left[R \ge \tfrac{1}{2} + |x|\right], \quad \forall |x| > 1/2.$$

If $|x| < 1/2$,

$$\mathbb{P}\left[R \le \tfrac{1}{2} - |x|\right] = 1 - \frac{1}{1+\left(\frac{\frac{1}{2}-|x|}{1-\frac{1}{2}+|x|}\right)^2} = 1 - \frac{1}{1+\left(\frac{\frac{1}{2}-|x|}{\frac{1}{2}+|x|}\right)^2} = 1 - \frac{\left(\frac{1}{2}+|x|\right)^2}{\left(\frac{1}{2}+|x|\right)^2+\left(\frac{1}{2}-|x|\right)^2}$$

and $$\mathbb{P}\left[R \ge \tfrac{1}{2} + |x|\right] = 1 - \mathbb{P}\left[R \le \tfrac{1}{2} + |x|\right]$$
$$= 1 - \left(1 - \frac{1}{1+\left(\frac{\frac{1}{2}+|x|}{1-\frac{1}{2}-|x|}\right)^2}\right)$$
$$= \frac{1}{1+\left(\frac{\frac{1}{2}+|x|}{\frac{1}{2}-|x|}\right)^2}$$
$$= \frac{\left(\frac{1}{2}-|x|\right)^2}{\left(\frac{1}{2}-|x|\right)^2+\left(\frac{1}{2}+|x|\right)^2}$$
$$= 1 - \frac{\left(\frac{1}{2}+|x|\right)^2}{\left(\frac{1}{2}+|x|\right)^2+\left(\frac{1}{2}-|x|\right)^2} = \mathbb{P}\left[R \le \tfrac{1}{2} - |x|\right]$$

So, $\mathbb{P}\left[R \le \tfrac{1}{2} - |x|\right] = \mathbb{P}\left[R \ge 1/2 + |x|\right] \quad \forall \ |x| \ge 0.$

Hence, the distribution of $R$ is symmetric about $\frac{1}{2}$.

$\therefore\ \mathbb{E}[R] = \frac{1}{2}$.

d) $\mathrm{Cov}[R, \sqrt{X}+\sqrt{Y}] = \mathbb{E}[R(\sqrt{X}+\sqrt{Y})] - \mathbb{E}[R].\mathbb{E}[\sqrt{X}+\sqrt{Y}]$

$= \mathbb{E}[R(\sqrt{X}+\sqrt{Y})] - \mathbb{E}[R]\cdot(\mathbb{E}[\sqrt{X}] + \mathbb{E}[\sqrt{Y}])$

Now, $\mathbb{E}[R(\sqrt{X}+\sqrt{Y})]$

$= \mathbb{E}[\ R(\sqrt{X}+\sqrt{Y}).1[X>0, Y>0] + R(\sqrt{X}+\sqrt{Y}).\ 1[\{X>0, Y>0\}^c]\ ]$

$= \mathbb{E}[\ R(\sqrt{X}+\sqrt{Y})\cdot 1[X>0, Y>0]\ ] + \mathbb{E}[\ R(\sqrt{X}+\sqrt{Y})\cdot 1[\{X>0, Y>0\}^c]\ ]$

$= \mathbb{E}[\ \frac{\sqrt{X}}{\sqrt{X}+\sqrt{Y}}\cdot(\sqrt{X}+\sqrt{Y}).\ 1[X>0].\ 1[Y>0]\ ] + \mathbb{E}[\ 0\cdot(\sqrt{X}+\sqrt{Y})\cdot\ 1[\{X>0, Y>0\}^c]\ ]$

$= \mathbb{E}[\ \sqrt{X}\cdot 1[X>0]\ 1[Y>0]\ ]$

$= \mathbb{E}[\ (\mathbb{E}[\ (1[Y>0]\ \sqrt{X}\ 1[X>0])\ |\ Y=y\ ])_{y=Y}\ ]$

$= \mathbb{E}[(\mathbb{E}[(1[y>0]\ \sqrt{X}.1[X>0])\ |\ Y=y])_{y=Y}]$

$= \mathbb{E}[\ (1[y>0].\mathbb{E}[\ \sqrt{X}.1[X>0]\ |\ Y=y\ ])_{y=Y}\ ]$

$= \mathbb{E}[(\ 1[y>0].\mathbb{E}[\ \sqrt{X}.\ 1[X>0]\ ])_{y=Y}\ ]$ $[\because X \perp\!\!\!\perp Y]$

$= \mathbb{E}[\ 1[Y>0]\ .\ \mathbb{E}[\ \sqrt{X}.1[X>0]\ ]\ ]$

$= \mathbb{E}[\ 1[Y>0]\ ].\ \mathbb{E}[\sqrt{X}.1[X>0]]$ [$\because$ The inner expectation is a constant]

$= \mathbb{P}[Y>0].\ \mathbb{E}[\ \sqrt{X}.\ 1[X>0]\ ]$

$= \mathbb{E}[\sqrt{X}.\ 1[X>0]]$ $[\because Y \sim \exp(\mu) \Rightarrow \mathbb{P}[Y>0]=1]$

$$= \int_0^\infty x^{1/2} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \mathbb{E}[\sqrt{X}].$$

$$\therefore \operatorname{Cov}[R, \sqrt{X}+\sqrt{Y}] = \mathbb{E}[\sqrt{X}] - \mathbb{E}[R]\cdot(\mathbb{E}[\sqrt{X}] + \mathbb{E}[\sqrt{Y}])$$

$$= \mathbb{E}[\sqrt{X}] - \tfrac{1}{2}(\mathbb{E}[\sqrt{X}] + \mathbb{E}[\sqrt{Y}])$$

$$= \tfrac{1}{2}(\mathbb{E}[\sqrt{X}] + \mathbb{E}[\sqrt{Y}]).$$

$$\mathbb{E}[\sqrt{X}] = \int_0^\infty x^{1/2} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty x^{\frac{3}{2}-1} e^{-\lambda x}\, dx$$

$$= \lambda \cdot \frac{\Gamma(\frac{3}{2})}{(\lambda)^{3/2}}$$

[ Gamma integral $\int_0^\infty e^{-\alpha x} \cdot x^{p-1}\, dx = \frac{\Gamma(p)}{\alpha^p}$, $p \in \mathbb{Q}$ (set of rationals) ]

$$= \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \frac{\sqrt{\pi}}{2\sqrt{\lambda}}.$$

Similarly, $\mathbb{E}[\sqrt{Y}] = \dfrac{\sqrt{\pi}}{2\sqrt{\mu}}$.

$$\therefore \operatorname{Cov}[R, \sqrt{X}+\sqrt{Y}] = \frac{1}{2}\left(\frac{\sqrt{\pi}}{2\sqrt{\lambda}} + \frac{\sqrt{\pi}}{2\sqrt{\mu}}\right)$$

$$= \frac{\sqrt{\pi}}{4}\left(\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\mu}}\right)$$

If $\lambda = \mu$,

$$\operatorname{Cov}[R, \sqrt{X}+\sqrt{Y}] = \frac{\sqrt{\pi}}{4} \cdot \frac{2}{\sqrt{\lambda}} = \frac{\sqrt{\pi}}{2\sqrt{\lambda}}.$$