CNET 324 - 01 * / Spring 2018

Homework # 1 : Solutions

① Done in the recitation.

② The solution is provided in the PDF available through http://www.athenasc.com/prob-solved-2ndedition.pdf

③ }
④ } Solutions are available in the Textbook.

5) Statement: S is finite (resp. countably infinite)
$\equiv$ S is equipotent with the set $\{1,\dots,n\}$ for some integers in $\mathbb{N}$ (resp. $\mathbb{N}$)

Proof:

($\Rightarrow$) Suppose S is finite (resp. countably infinite)

Then, $\exists$ a one-one mapping $g: S \to \mathbb{N}$. ——— (i)

So, with $a, a' \in S$, $g(a) = g(a') \Rightarrow a = a'$; $g(a), g(a') \in \mathbb{N}$.

As $\{1,\dots,n\} \subset \mathbb{N}$, with $n \in \mathbb{N}$,
with $a, a' \in S$, $g(a) = g(a') \Rightarrow a = a'$ for $g(a), g(a') \in \{1,\dots,n\}$

So, $g: S \to \{1,\dots,n\}$ is one-one. ——— (ii)

Since S is finite, $g(S)$ has exactly n elements by definition.
(Note: $g(S) = \{g(a): a \in S\}$)

So, $g(S) = \{1,\dots,n\}$ for some integers in $\mathbb{N}$.

So, $g: S \to \{1,\dots,n\}$ is onto. ——— (iii)

From (ii) & (iii), $g: S \to \{1,\dots,n\}$ is bijection.

$\Rightarrow$ S is equipotent with the set $\{1,\dots,n\}$.

(resp. if S is countably infinite, for any one-one mapping $g: S \to \mathbb{N}$ (which exists, (i)), there is no $n \in \mathbb{N}$ s.t. $g(S)$ has exactly n elements.

Then $g(S) = \mathbb{N}$. ——— (iv)

From (i) & (iv), $g: S \to \mathbb{N}$ is a bijection

$\Rightarrow$ S is equipotent with the set $\mathbb{N}$.)

($\Leftarrow$)

Suppose there is a bijection $g: S \to \{1,\dots,n\}$ for some integers in $\mathbb{N}$, i.e., S is equipotent with the set $\{1,\dots,n\}$.

So, $g: S \to \{1,\dots,n\}$ is one-one ——— (i)

and $g(S) = \{1,\dots,n\}$ (onto) ——— (ii)

As $\{1,\dots,n\} \subset \mathbb{N}$, $g : S \to \mathbb{N}$ is one-one also if $g : S \to \{1,\dots,n\}$ is one-one, with $g(S) = \{1,\dots,n\}$ being the image of S in $\mathbb{N}$. So, S is countable. ——— (iii)

Also, $g(S) = \{1,\dots,n\}$ for some integers in $\mathbb{N}$
$\Rightarrow$ $g(S)$ has exactly n distinct elements ——— (iv)

From (iii) & (iv), S is finite.

(resp. if S is equipotent with $\mathbb{N}$ itself,

$\exists$ a bijection $g : S \to \mathbb{N}$
$\Rightarrow$ $g : S \to \mathbb{N}$ is one-one and $g(S) = \mathbb{N}$.

Then, for a one-one mapping $g : S \to \mathbb{N}$, we have $g(S) = \mathbb{N}$. So, we cannot find some $n \in \mathbb{N}$ s.t. $g(S)$ will have exactly n elements, because $|g(S)| = |\mathbb{N}|$ which is not a finite number.

So, S is countably infinite.)

We have shown that each side of the statement implies the other side. Hence, they are equivalent.

6.a) Proof:

S is a countable set.

$\Rightarrow \exists$ a one-one mapping $f: S \to \mathbb{N}$

Take any subset $R \subseteq S$

Define a mapping $g: R \to S$ as $g(a) = a, \forall a \in R$

Clearly g is one-one, as it maps to itself.

Now, consider the composition function $f \circ g : R \to \mathbb{N}$.

(Note: $(f \circ g)(a) = f(g(a))$, $a \in R$)

$f \circ g$ is composition of two one-one mappings. Hence, $f \circ g$ is one-one and $f \circ g: R \to \mathbb{N}$.

So, R is countable. (Proved)

6.b) The statement is not true.

Choose S as $\mathbb{R}$ (set of real numbers) and R as $\mathbb{N}$.

Then $R \subset S$.

Here $S = \mathbb{R}$ is not countable, but $R = \mathbb{N}$ is.

7) S is countable set and finite.

$f: S \to \mathbb{N}$ and $g: S \to \mathbb{N}$ are two one-one mappings

$f(S)$ has exactly n_f elements and $g(S)$ has exactly n_g elements.

(Note: $f(S) = \{ f(x) : x \in S \}$)

We need to show that $n_f = n_g$.

Proof:

Since $f: S \to \mathbb{N}$ is one-one, each element of $f(S)$ is mapped from at most one element in S.

Also, each element of $f(S) = \{ f(x) : x \in S \}$ has a pre-image in S (by definition of $f(S)$)

From these two statement, we can conclude that S has n_f elements.

Similarly, for any one-one mapping $g: S \to \mathbb{N}$, with $g(S)$ having n_g elements, S has exactly n_g elements.

$\therefore n_f = n_g$, considering the number of distinct elements in S.

(Proved)

(Thus, for a countable finite set S, there can be more than one one-to-one mapping from S to $\mathbb{N}$, but the image of S in $\mathbb{N}$ will have same number of elements in all those mappings, which is the cardinality of the set S. Hence, we can enumerate the elements of S as $\{ a_1, \ldots, a_n \}$, n being the cardinality of S.)

(8) Fact 1 : If $f : A \to B$ and $g : B \to C$ are bijections, $h = (g \circ f)$ $A \to C$ is also a bijection.

Fact 2 : If A and B are equipotent, there exists bijections $g : A \to B$ and $h : B \to A$.

(8.a) (a) If A is finite, $\exists$ a bijection $f_1 : A \to C_1$, $C_1 = \{1, \dots, n\}$ for some integer in $\mathbb{N}$.

$\therefore$ $f_1 \circ h : B \to C_1$ is also a bijection (From fact 1 & 2), $C_1 = \{1, \dots, n_1\}$ for some integer in $\mathbb{N}$.

$\therefore$ By definition, B is also finite.

(b) Similarly, if B is finite, $\exists$ a bijection $f_2 : B \to C_2$, $C_2 = \{1, \dots, m\}$ for some integer in $\mathbb{N}$.

$\therefore$ $f_2 \circ g : A \to C_2$ is also a bijection (From fact 1 & 2), $C_2 = \{1, \dots, m\}$ for some integer in $\mathbb{N}$.

$\therefore$ A is also finite.

Hence, A is finite iff B is finite. //

(8.b) (a) If A is countably infinite, $\exists$ a bijection $f_1 : A \to \mathbb{N}$ (From fact 1 & 2) $f_1 \circ h : B \to \mathbb{N}$ will also be a bijection

$\therefore$ By definition, B is also countably infinite.

(b) Similarly, if B is countably infinite, $\exists$ a bijection $f_2 : B \to \mathbb{N}$

Then, $f_2 \circ g : A \to \mathbb{N}$ is also a bijection (From Fact 1 & 2)

$\therefore$ A is also countably infinite.

Thus, A is countably infinite iff B is countably infinite //

(8.c) (a) If A is uncountable, A is not countable. Assume that B is countable (The approach is proof by contradiction.).

Then, $\exists$ $f_1 : B \to \mathbb{N}$, a bijection.

$\therefore$ $f_1 \circ g : A \to \mathbb{N}$ is also a bijection (Fact 1 & 2), which implies that A is countable. But this is a contradiction. Hence, the assumption is incorrect. $\therefore$ B is uncountable.

(b) Similarly, suppose B is uncountable. Assume A is countable. Then, $\exists$ $f_2 : A \to \mathbb{N}$, a bijection $\Rightarrow$ $f_2 \circ h : B \to \mathbb{N}$ is also a bijection, implying that B is countable. But this is a contradiction. Hence, the assumption is incorrect $\Rightarrow$ A is uncountable.

Thus, A is uncountable iff B is uncountable. //

(9) Theorem : If there exists injective functions $g: A \to B$ and $h: B \to A$ between the sets A and B, then there exists a bijective function $f: A \to B$.

According to the given problem, there $\exists$ injective mappings $g: \mathbb{N} \to A$ and $h: A \to \mathbb{N}$. Hence, according to the above theorem, $\exists$ a bijective mapping $f: A \to \mathbb{N}$.

$\therefore$ A is countably infinite. //

(10) Assume that the 3 classes are distinct.

$\square \quad \square \quad \square$
$c1 \quad c2 \quad c3$

Without any restrictions,

there are $\binom{90}{30}$ ways of picking students to C1, $\binom{60}{30}$ ways to C_2 and $\binom{30}{30}$ ways to C_3.

$\therefore$ Total # of possible grouping arrangements $= \binom{90}{30}\binom{60}{30}\binom{30}{30}$

$$= \frac{90!}{(30!)^3} \quad \text{—— (1)}$$

Now, consider the cases where Joe and Jane are in the same class. They can be in any of the 3 classes, and for that class there are $\binom{88}{28}$ ways of picking the other students. For the two remaining classes, the # of ways of picking students are $\binom{60}{30}$ and $\binom{30}{30}$ respectively.

$\therefore$ # of groups s.t. Joe & Jane end up in the same class $\Big\} = 3 \times \binom{88}{28} \times \binom{60}{30} \times \binom{30}{30}$

$$= \frac{3 \times 88!}{28! \times 30! \times 30!} \quad \text{—— (2)}$$

$\therefore$ The probability that Joe & Jane end up in the same class $\Big\}$ $= (2) / (1)$

$$= \frac{3 \times 88!}{28! \times 90!} \times 30!$$

$$= \frac{3 \times 30 \times 29}{90 \times 89}$$

$$= \frac{29}{89}$$

$$= 0.326 \;//$$