ENEE 324-01 / SPRING 2108

Homework # 2 (Solutions)

(1) This is a problem of circular permutation.

(1.a) First, consider the problem of seating $(M+W)$ people in $(M+W)$ seats (when not in a circle). When the seats are distinct, this simply a counting problem with order and without replacement. Hence, there are $(M+W)!$ possible arrangements.

Now consider the fact that these $(M+W)$ seats are on a circle. Thus, given an arrangement, if you rotate it by any n $(0 < n \leq M+W)$, you get the same arrangement (i.e. the same people are still on the right and left of each person, circular shifting doesn't give a new configuration). Therefore, out of the previously counted $(M+W)!$ arrangements, each arrangement is repeatively include $(M+W)$ times

$\therefore$ total # of possible seating arrangements $= \dfrac{(M+W)!}{(M+W)}$

$= (M+W-1)!$ //

(1.b) Given that all women are seated contiguosly, all men are also seated contiguously. Hence, can consider the group of women as a single entity and the group of men as another single entity. The number of ways to place two entities on a circle is 1 (Same argument as before: $2!/2 = 1$). But, there are $M!$ distinct arrangements among men and $W!$ distinct arrangements among women.

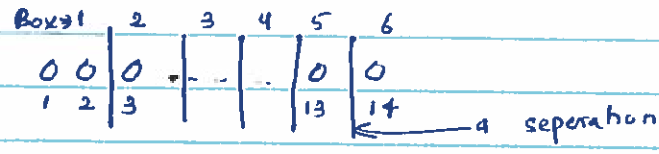
$\therefore$ total # of arrangements $= 1 \times M! \times W!$

$= M!\,W!$ //

② }
Solutions available through the link mentioned in HW
③ }

④ Solution available in the textbook.

(5) Assume that the balls are identical. (In case they are also distinct, the number of possible groupings is simply 6^{14}). Suppose the balls are arranged linearly as follows.



The problem of grouping these to 6 distinct boxes is anologous to the problem of placing 5 seperators as indicated in the diagram. Assuming that one or more boxes can also be empty (since it is not specified in the problem), this has now become a problem of choosing 5 locations (for the seperations), out of the $(5+14)$ total number of locations (seperations + balls).

$\therefore$ Total # of distinct configurations $= \binom{19}{5}$ //

Now, suppose we add the constraint 'there should be at-least one ball in each box'. Again consider a linear arrangement of the 14 balls.

Since there should be at-least one ball in each box, now, the total number of locations available for the seperators is 13 (the 13 gaps indicated by arrows) Hence, this become a problem of choosing 5 locations (for the seperators) out of 13.

$\therefore$ Total # of distinct configurations $= \binom{13}{5}$ //

(5) Each person has 365 possibilities for the birthday. Hence, for k passengers, total # of possible combinations $= 365 \times 365 \times \ldots \times 365$ (k times)

$$= 365^k$$

(6.a) Assume $1 \leq k \leq 365$.

$$\# \text{ of possible combinations} = 365 \times 364 \times \ldots \times (365 - k + 1)$$

$$= \frac{365!}{(365-k)!}$$

(a problem of ordering without replacement)

(6.b) Assume $2 < k \leq 365$

$$\left.\begin{array}{l}\# \text{ of ways to pick 2 passengers out} \\ \text{of } k\end{array}\right\} = \binom{k}{2}$$

The b'day is fixed as Jan 1 for these 2 passengers. For the remaining $(k-2)$, can pick out of the other 364 days, but without repetition.

$$\therefore \# \text{ of ways to the remaining } (k-2) = 364 \times 363 \times \ldots \times (367-k)$$

$$= \frac{364!}{(364-(k-2))!}$$

$$\therefore \text{total \# of possible lists} = \binom{k}{2} \times \frac{364!}{(366-k)!}$$

(7) No. This is not always true!

Counter - Example :

Suppose $\Omega = \mathbb{N}$ and $A_k = \{k, k+1, \ldots\}$, $k = 1, 2, 3, \ldots$

(i.e. $A_k = \{\omega \in \Omega \mid \omega \geqslant k\}$, $k = 1, 2, 3, \ldots$)

For each $n = 1, 2, \ldots$

$$\bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \ldots \cap A_n$$

$$= A_n \quad (\because A_1 \supset A_2 \supset \ldots \supset A_n)$$

$$= \{\omega \in \Omega \mid \omega \geqslant n\} \neq \phi$$

Hence, this satisfies the condition given in the problem.

But, $\bigcap_{k=1}^{\infty} A_k$ is empty.

Why?

(Proof by contradiction. Suppose there exists atleast one element (m) in $\bigcap_{k=1}^{\infty} A_k = A$. i.e. $\exists A_m$, $m \in \mathbb{N}$ s.t. $m \in A$ is also, $m \in A_1$, $m \in A_2$, ... $m \in A_m$, But $m \notin A_{m+1}$, hence $m \notin \bigcap_{k=1}^{m+1} \Rightarrow m \notin A$, which is a contradiction.

Hence, $\bigcap_{k=1}^{\infty} A_k$ does not contain any elements $\Rightarrow$ is empty)

- Whatever $n \in \Omega$ you pick, the sets after A_{n+1} (i.e. $A_{n+1}, A_{n+2}, \ldots$) will not contain n. Hence, n will not be in $\bigcap_{k=1}^{\infty} A_k$. -

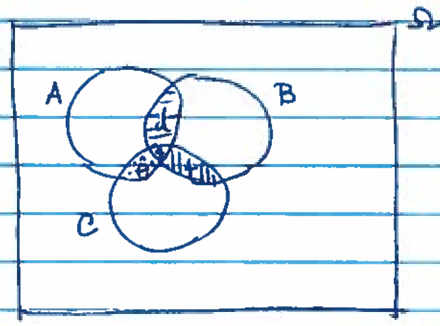
8> 3-digit even numbers with digits from $\{1,5,6,8\}$ without any repeating digit.

Since the number has to be even, unit's place has 2 choices (6 or 8). For each such choice, ten's place has 3 choices (from the other 3 digits in that set except the one already in the unit's place), and in a similar way, for each such choice, the hundred's place has rest of the 2 choices (except the twos already used in ten's & unit's place).

So, possible ways are $2 \times 3 \times 2 = 12$.

9> Let A: set of people who are registered Whig
B: set of people who approve Fillmore's performance
C: set of people who favor the Electoral College system.

Then,



From the information,

$$|\Omega| = 1000$$
$$|C| = 550$$
$$|g| = 100$$
$$|d| + |g| = 125 \quad \Rightarrow |d| = 25$$
$$|d| + |e| + |f| = 325 \quad \Rightarrow |e| + |f| = 300$$

Required set has cardinality $= |C| - |f| - |g| - |e|$
$= 550 - (|e| + |f|) - 100$
$= 550 - 300 - 100$
$= 150.$

10.a) 20 T/F questions

Half of these, i.e., 10 questions can be answered T from a set of 20 questions in $\binom{20}{10}$ ways. Rest of the 10 can be answered F in $\binom{10}{10}$ way.

So, total possibilities to answer in such a way is $\binom{20}{10} \times \binom{10}{10}$

$$= \frac{20!}{10!\,10!}$$

10.b) There are only 2 choices (T/F) for each question and no two consecutive choices are same.

There is only 2 possible patterns for that:

T F T F ...... T F

and F T F T .... . F T

(1 2 3 4 .. 19 20)