— ENEE 324-01* / Spring 2018
— Homework #3 Solutions

1) $(\Omega, \mathcal{F}, \mathbb{P})$

$A_1, A_2, \ldots, A_n \in \mathcal{F}$

To prove : $\mathbb{P}[\bigcup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \mathbb{P}[A_i]$ ——— (i)

Proof: Use induction.

For $n=2$,

$$\mathbb{P}[\bigcup_{i=1}^{2} A_i] = \mathbb{P}[A_1 \cup A_2]$$

$$= \mathbb{P}[A_1] + \mathbb{P}[A_2] - \mathbb{P}[A_1 \cap A_2]$$

$$\leq \mathbb{P}[A_1] + \mathbb{P}[A_2] \quad [\because \mathbb{P}[A_1 \cap A_2] \geq 0 \text{ by definition of } \mathbb{P}]$$

Now suppose (i) holds for $n=m$, i.e.,

$$\mathbb{P}[\bigcup_{i=1}^{m} A_i] \leq \sum_{i=1}^{m} \mathbb{P}[A_i] \text{ ——— (ii)}$$

We have to show, (i) holds for $n=m+1$ also.

$$\mathbb{P}[\bigcup_{i=1}^{m+1} A_i] = \mathbb{P}[\bigcup_{i=1}^{m} A_i \cup A_{m+1}]$$

$$= \mathbb{P}[\bigcup_{i=1}^{m} A_i] + \mathbb{P}[A_{m+1}] - \mathbb{P}[(\bigcup_{i=1}^{m} A_i) \cap A_{m+1}]$$

$$\leq \mathbb{P}[\bigcup_{i=1}^{m} A_i] + \mathbb{P}[A_{m+1}] \quad [\because \mathbb{P}[(\bigcup_{i=1}^{m} A_i) \cap A_{m+1}] \geq 0]$$

$$\leq \sum_{i=1}^{m} \mathbb{P}[A_i] + \mathbb{P}[A_{m+1}] \quad [\text{from (ii)}]$$

$$= \sum_{i=1}^{m+1} \mathbb{P}[A_i]$$

i.e., (i) holds for $n=m+1$.

Then, by induction, (i) holds for $n = 2, 3, \ldots$ (Proved)

2) } 3) } 4) } 5) } 6) } Solution available through the link mentioned in HW.

7) The sample space is all possible permutations of 52 cards (distinct cards)

So, $\Omega = \{(a_1, \dots, a_{52}) : a_i \neq a_j \text{ for } i \neq j,\ a_i \in \text{the set of 52 cards}\}$ $j, i = 1, \dots, 52$

Since, $|\Omega| < \infty$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

Since the deck is well shuffled,

$$\mathbb{P}[\{\omega\}] = \frac{1}{|\Omega|} = \frac{1}{52!} = p, \ \forall \omega \in \Omega$$

At this point, we have defined the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$.
Now we go on to the prob. of the said event A: the 13^{th} card is the first King to be dealt.

The event A requires: the first 12 cards are non-King (a)
and
the 13^{th} card is a King (b)
and
the last ~~4~~ 39 cards can be anything from the remaining deck. (c)

(a) has $\binom{48}{12} \times 12!$ arrangements [48 non-King cards, from that select any 12 and permute them in any order]

(b) has 4 choices. [one of the 4 Kings]

(c) has $39!$ arrangements possible.

So total number of ω's where event A happens is

$$|A| = \binom{48}{12} 12! \times 4 \times 39!$$

$\therefore \mathbb{P}[A] = p \times |A|$

$$= \frac{1}{52!} \times \binom{48}{12} \times 12! \times 4 \times 39!$$

$$= \frac{48 \times \cdots \times 37 \times 4}{52 \times \cdots \times 40}$$

8) The sample space is all possible permutations of 20 distinct cars.

So, $\Omega = \{(a_1, \ldots, a_{20}) = \omega : a_i \neq a_j \text{ for } i \neq j, a_i \text{ is a car}, i, j = 1, \ldots, 20\}$

Since $|\Omega| < \infty$,

$$\mathcal{F} = \mathcal{P}(\Omega)$$

Since the position any car take is random,

$$\mathbb{P}[\{\omega\}] = \frac{1}{|\Omega|} = \frac{1}{20!} = p, \quad \forall \omega \in \Omega.$$

a) The number of possible way 20 cars can line up in the parking lot with 20 places is

$$|\Omega| = 20!$$

b) Event A: no two US cars are adjacent and no two foreign cars are adjacent

To find the number of ω's where A happens:

If the first one is US car, it ~~has 10~~ could be one of the 10 US cars. Now, the second one has to be a foreign car, which could be one of the 10 foreign cars. The 3rd one has to be one of the remaining 9 US cars, and so on.

So, possible arrangements is $10 \times 10 \times 9 \times 9 \ldots \times 1 \times 1 = 10! \times 10!$

If the first one is foreign car, that will also have $10! \times 10!$ arrangeme
in a similar way as before.

So, $|A| = 2 \times 10! \times 10!$

$\therefore P[A] = \cancel{} p \times |A|$

$$= \frac{2 \times 10! \times 10!}{20!}$$

9) } Solution available through the link mentioned in HW set.
10) }