ENEE 324-01 / SPRING 2018

HOMEWORK # 5

SOLUTIONS

(1) Given that $A_1, A_2, \ldots, A_n$ are Mutually Independent,

$$\mathbb{P}\left[\bigcap_{i \in S} A_i\right] = \prod_{i \in S} \mathbb{P}(A_i), \text{ for } \underline{\text{any}}\ S \subseteq \{1, 2, \ldots, n\}, \quad |S| \geq 2 \qquad \text{—— (1)}$$

Now consider any $I \subseteq \{1, 2, \ldots, n\}$ with $|I| \geq 2$,

Consider any subset J of I ($J \subseteq I$) such that $|J| \geq 2$.

But since, $J \subseteq I \subseteq \{1, 2, \ldots, n\}$, $J \subseteq \{1, 2, \ldots, n\}$ as well, such that $|J| \geq 2$.

Hence, from (1);

$$\mathbb{P}\left[\bigcap_{i \in J} A_i\right] = \prod_{i \in J} \mathbb{P}(A_i)$$

As this is true for $\underline{\text{any}}$ $J \subseteq I$ s.t. $|J| \geq 2$, the set of events $\{A_i, i \in I\}$ are Mutually Independent.

Since I is also arbitrary, for any subset I of $\{1, \ldots, n\}$ with $|I| \geq 2$, the events $\{A_i, i \in I\}$ are Mutually Independent.

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(2) The events A, B, C and D are mutually independent.

Hence, the events obtained by applying basic set theoretic operations on disjoint partitions of the set $\{A, B, C, D\}$ are also Mutually Independent.

By using this theorem, we can concluded that the 4 cases 2.a., 2.b., 2.d., and 2.e. are Mutually Independent whereas 2.c. is not. Can mathematically prove that as illustrated by the following example.

(2a.) $\mathbb{P}\left[(A \cup B^c) \cap C\right]$

$= \mathbb{P}\left[(A \cap C) \cup (B^c \cap C)\right]$

$= \mathbb{P}(A \cap C) + \mathbb{P}(B^c \cap C) - \mathbb{P}(A \cap C \cap B^c \cap C)$

$= \mathbb{P}(A).\mathbb{P}(C) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B^c \cap C)$

$= \mathbb{P}(A).\mathbb{P}(C) + \mathbb{P}(C) - \mathbb{P}(B).\mathbb{P}(C) - \left[\mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C)\right]$

$= \mathbb{P}(A)\,\mathbb{P}(C) + \mathbb{P}(C) - \mathbb{P}(B).\mathbb{P}(C) - \mathbb{P}(A).\mathbb{P}(C) + \mathbb{P}(A).\mathbb{P}(B).\mathbb{P}(C)$

$= \mathbb{P}(C)\left[\mathbb{P}(A) + 1 - \mathbb{P}(B) - \mathbb{P}(A)\left(1 - \mathbb{P}(B)\right)\right]$

$= \mathbb{P}(C)\left[\mathbb{P}(A) + \mathbb{P}(B^c) - \left(\mathbb{P}(A) - \mathbb{P}(A \cap B)\right)\right]$

$= \mathbb{P}(C)\left[\mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c)\right]$

$= \mathbb{P}(C)\,\mathbb{P}\left[A \cup B^c\right]$

$\Rightarrow (A \cup B^c) \perp\!\!\!\perp C$ // (Try b, d, e similarly)

(2.c) $\mathbb{P}\left[(A \cup B) \cap B\right]$

$= \mathbb{P}\left[(A \cap B) \cup B\right]$

$= \mathbb{P}(A \cap B) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$= \mathbb{P}(B)$

$\neq \mathbb{P}(A \cup B)\,.\,\mathbb{P}(B)$

$\Rightarrow (A \cup B) \not\perp\!\!\!\perp B$ ($(A \cup B)$ & B are not M.I.) //

3)

4) Solution available in the link mentioned earlier.

5)

6)

7) Done in Recitation 5.

8) Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ be the probability triple for $\mathcal{E}_i$, $i=1,\dots,n$.

$\mathbb{P}$ is uniform probability measure on $\Omega = \Omega_1 \times \dots \times \Omega_n$.

i) Consider a singleton $\{\omega_i\} \subseteq \Omega_i$, $i=1,\dots,n$

Then, $\{\omega_i\} = \Omega_1 \times \dots \times \Omega_{i-1} \times \{\omega_i\} \times \Omega_{i+1} \times \dots \times \Omega_n \subseteq \Omega$

$\Rightarrow \mathbb{P}[\{\omega_i\}] = \mathbb{P}[\Omega_1 \times \dots \times \Omega_{i-1} \times \{\omega_i\} \times \Omega_{i+1} \times \dots \times \Omega_n]$

Now, $\Omega = \Omega_1 \times \dots \times \Omega_n$ where all Ω_i's are finite.

So, $|\Omega| = |\Omega_1 \times \dots \times \Omega_n| = |\Omega_1| \times \dots \times |\Omega_n|$ [Cardinality of cartesian products of finite sets is the product of their respective cardinalities]

$< \infty$ [Since, Ω_i's are finite]

Similarly, $|\Omega_1 \times \dots \times \Omega_{i-1} \times \{\omega_i\} \times \Omega_{i+1} \times \dots \times \Omega_n|$

$= |\Omega_1| \times \dots \times |\Omega_{i-1}| \times |\{\omega_i\}| \times \dots \times |\Omega_{i+1}| \times \dots |\Omega_n|$

$\therefore \mathbb{P}[\{\omega_i\}] = \dfrac{|\Omega_1| \times \dots \times |\Omega_{i-1}| \times |\{\omega_i\}| \times \dots \times |\Omega_{i+1}| \times \dots \times |\Omega_n|}{|\Omega_1| \times \dots \times |\Omega_n|}$ [Since, $\mathbb{P}$ is uniform on Ω]

$= \dfrac{|\{\omega_i\}|}{|\Omega_i|}$

$= \dfrac{1}{|\Omega_i|}$, $\forall\, \omega_i \in \Omega_i$, $i=1,\dots,n$. —— (i)

Thus, $\mathbb{P}_i[\{\omega_i\}] = \mathbb{P}[\{\omega_i\}] = \dfrac{1}{|\Omega_i|}$ [$\because \mathbb{P}[\{\omega_i\}]$ is uniform $\forall\, \omega_i \in \Omega_i$, $i=1,\dots,n$]

Next, consider $B_i \in \mathcal{F}_i$ (i.e., $B_i \subseteq \Omega_i$), $i=1,2,\dots,n$

Since $\mathbb{P}_i$ is uniform in Ω_i,

$$\mathbb{P}_i[B_i] = \frac{|B_i|}{|\Omega_i|} \quad \text{—— (ii)}$$

$$= \frac{|\Omega_1||\Omega_2|\cdots|\Omega_{i-1}||B_i||\Omega_{i+1}|\cdots|\Omega_n|}{|\Omega_1||\Omega_2|\cdots|\Omega_{i-1}||\Omega_i||\Omega_{i+1}|\cdots|\Omega_n|}$$

$$= \frac{|\Omega_1 \times \cdots \Omega_{i-1} \times B_i \times \cdots \times \Omega_{i+1} \times \cdots \times \Omega_n|}{|\Omega_1 \times \cdots \times \Omega_i \times \Omega_{i+1} \times \cdots \times \Omega_n|}$$

$$= \frac{|A_i|}{|\Omega|}$$

$$= \mathbb{P}[A_i]$$

where A_i is, as defined in problem(7),

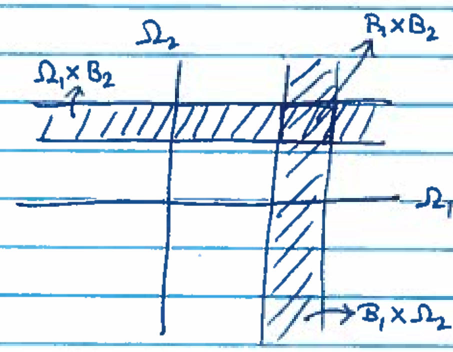
$$A_i = (\Omega_1 \times \cdots \times \Omega_{i-1} \times B_i \times \Omega_{i+1} \times \cdots \times \Omega_n) \in \mathcal{F}$$

and $B_i \in \mathcal{F}_i$, $i = 1, \ldots, n$

ii) Now, consider $(A_1 \cap \ldots \cap A_n)$.

For simplicity, choose $n=2$ for now.

Then we have $A_1 \cap A_2$



$$= (B_1 \times \Omega_2) \cap (\Omega_1 \times B_2)$$

$$= B_1 \times B_2$$

$$\therefore \mathbb{P}[A_1 \cap A_2] = \mathbb{P}[B_1 \times B_2]$$

$$= \frac{|B_1 \times B_2|}{|\Omega_1 \times \Omega_2|} \quad [\because B_1 \times B_2 \subset \Omega \text{ and } \mathbb{P} \text{ is uniform on } \Omega]$$

$$= \frac{|B_1||B_2|}{|\Omega_1||\Omega_2|}$$

$$= \frac{|B_1|}{|\Omega_1|} \times \frac{|B_2|}{|\Omega_2|}$$

$$= \mathbb{P}_1[B_1] \cdot \mathbb{P}_2[B_2] \text{ , for } n=2. \quad [\text{from (ii)}]$$

Similarly it can be shown that

$$\mathbb{P}[A_1 \cap \ldots \cap A_n] = \prod_{i=1}^{n} \mathbb{P}_i[B_i],$$

which is eq^{n} (1.1) in problem(7).

Hence, our model in part (i) is compatible with the $\mathcal{E}_i$'s being mutually independent, in sense of problem (7).

9. a) $\Omega = \{0,1\}^{\mathbb{N}_0}$ $[1 \equiv \text{Head}, \; 0 \equiv \text{Tail}]$

$= \{\omega = (\omega_1, \omega_2, \ldots) : \omega_k \in \{0,1\}, k = 1, 2, \ldots\}$

Consider $A_k \triangleq \{\omega \in \Omega : \omega_k = 1\}$, i.e., k^{th} toss is a Head.

Under independent & identical (iid) condition, $\{A_k : k = 1, 2, \ldots\}$ are mutually independent, and we can assign

$$\mathbb{P}[A_k] = p$$

$$\text{and } \mathbb{P}[A_k^c] = 1 - p$$

where $A_k^c = \{\omega \in \Omega : \omega_k = 0\}$, i.e., k^{th} toss is a Tail.

Now, for $n = 1, 2, \ldots$ and $b_k \in \{0,1\}$, consider the event

$$F_n(b_1, \ldots, b_n) := \{\omega \in \Omega : \omega_k = b_k, k = 1, \ldots, n\}.$$

$$\mathbb{P}[F_n(b_1, \ldots, b_n)] = \mathbb{P}\left[\bigcap_{k=1}^{n} \tilde{A}_k\right],$$

$$\text{where } \tilde{A}_k = \begin{cases} A_k, & \text{if } b_k = 1 \\ A_k^c, & \text{if } b_k = 0 \end{cases}$$

Since $\{A_k ; k = 1, 2, \ldots\}$ are mutually independent, $\{\tilde{A}_k : k = 1, \ldots\}$ are also mutually independent. [See the theorem in problem 1 of recitation 4]

$\therefore \mathbb{P}[F_n(b_1,\ldots,b_n)] = \mathbb{P}[\bigcap_{k=1}^{n} \tilde{A}_k]$

$= \prod_{k=1}^{n} \mathbb{P}[\tilde{A}_k]$ [by mutual ind.]

$= p^{\sum_{k=1}^{n} b_k} \cdot (1-p)^{\sum_{k=1}^{n}(1-b_k)}$, $n=1,2,\ldots$ [Since, $\mathbb{P}[\tilde{A}_k] = \begin{cases} p & \text{, if } b_k=1 \\ 1-p & \text{, if } b_k=0 \end{cases}$]

——(i)

b) $B = \{$a Tail eventually occurs$\}$

$= \bigcup_{n=1}^{\infty} F_n(\tilde{b}_n)$, [Under the notations defined in the model in part (a)]

where $\tilde{b}_n = (\underset{b_1}{1}, \underset{b_2}{1}, \ldots, 1, \underset{b_n}{0}) \in \{0,1\}^n$, $n=1,2,\ldots$

Under iid conditions, $\mathbb{P}[F_n(\tilde{b}_n)] = p^{n-1} \cdot (1-p)$ [The first $(n-1)$ tosses give Heads and the n^{th} one is Tail]

Since, $\{$getting the 1st Tail in k^{th} toss$\} = F_k(\tilde{b}_k)$ and $\{$getting the 1st Tail in m^{th} toss$\} = F_m(\tilde{b}_m)$ are disjoint $\forall\, l \neq k$,

$$\mathbb{P}[B] = \mathbb{P}[\bigcup_{n=1}^{\infty} F_n(\tilde{b}_n)]$$

$$= \sum_{n=1}^{\infty} \mathbb{P}[F_n(\tilde{b}_n)]$$

$$= \sum_{n=1}^{\infty} \mathbb{P}[F_n(1,1,\ldots,1,0)]$$

$$= \sum_{n=1}^{\infty} p^{n-1} \cdot (1-p) \quad \text{[using the prob. model (i)]}$$

$$= (1-p) \cdot \frac{1}{1-p}$$

$$= 1.$$

Hence, the prob. of never getting a tail is $\mathbb{P}[B^c] = 0$.

c) Here we are looking for sequences like

$$0,1,\ldots$$
$$0,0,1,\ldots$$
$$0,0,0,1,\ldots$$
$$\vdots$$

so that at least one Tail comes before a Head.
Note that, each such event is disjoint with any other of them.

$C = \{$getting Tail before even getting a ~~Tai~~ Head$\}$

$$= \bigcup_{n=2}^{\infty} F_n(\tilde{c}_n),$$

where $\tilde{c}_n = (0,0,\ldots,0,1) \in \{0,1\}^n$, $n = 2,3,\ldots$

$$\therefore \mathbb{P}[C] = \mathbb{P}\Big[\bigcup_{n=2}^{\infty} F_n(\tilde{c}_n)\Big] \quad \text{[Since, disjoint events]}$$

$$= \sum_{n=2}^{\infty} \mathbb{P}[F_n(\tilde{c}_n)]$$

$$= \sum_{n=2}^{\infty} \mathbb{P}[F_n(\underbrace{0,0,\ldots,0}_{n-1},1)]$$

$$= \sum_{n=2}^{\infty} (1-p)^{n-1}\cdot p \quad \text{[using (i)]}$$

$$= p\sum_{n=2}^{\infty} (1-p)^{n-1}$$

$$= p\cdot\frac{(1-p)}{1-(1-p)}$$

$$= (1-p).$$

10) $f: X \to Y$

$g: Y \to Z$

$h: X \to Z$ given by $h(x) = g(f(x)), \forall x \in X$

Here we assume, f^{-1} & g^{-1} exists.

Now, consider $z \in F, \forall F \subseteq Z$.

If f^{-1} & g^{-1} exists, f & g are bijection.

Hence, their composition $h = g \circ f$ is also a bijection.

So, h^{-1} exists.

Now, $h(h^{-1}(z)) = z$ [by def^n of h^{-1}]

$\Rightarrow g(f(h^{-1}(z)) = z$ [$\because h(x) = g(f(x))$]

$\Rightarrow f(h^{-1}(z)) = g^{-1}(z)$ [$\because g^{-1}$ exists]

$\Rightarrow h^{-1}(z) = f^{-1}(g^{-1}(z))$ [$\because f^{-1}$ exists], $\forall z \in F$

$\therefore h^{-1}(F) = f^{-1}(g^{-1}(F)), \forall F \subseteq Z$. (Proved).