ENEE 324-01 / SPRING 2018

HOMEWORK # 6

Solutions.

Problems 1 – 8 ⇒ Solutions are available in the solution manual!

9) HW#6 / Problem 9.

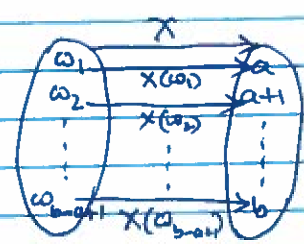
Sol^n:

The given pmf $\{p_{a,b}(x), x = a, a+1, \dots, b\}$ is uniform.

We have to think about a probability triple $(\Omega, \mathcal{A}, \mathbb{P})$ and an associated r.v. X whose pmf is $\{p_{a,b}(x), x = a, \dots, b\}$ under that $\mathbb{P}$ we have defined.

Idea: If we have a random experiment $\mathcal{E}$ with finite Ω of cardinality $(b-a+1)$, which is the number of values that x can take in the interval $\{a, a+1, \dots, b\}$ where $p_{a,b}(x)$ is defined

Next, to have
$$\mathbb{P}[X=x] = p_{a,b}(x), \quad x \in \{a, a+1, \dots, b\},$$
we can define a one-one mapping $X: \Omega \to \{a, \dots, b\}$
s.t. $\forall \omega \in \Omega$,
$$\mathbb{P}[\{\omega\}] = \mathbb{P}[\omega \in \Omega : X(\omega) = x], \forall x \in \{a, \dots,$$
$$= \mathbb{P}[X = x]$$



Now we construct such an $\mathcal{E}$, $(\Omega, \mathcal{A}, \mathbb{P})$ and X.

$\mathcal{E}$: Rolling a fair $(b-a+1)$ sided die once.
Then, $\Omega = \{1, 2, \dots, b-a+1\}$ $(a \le b)$
Since $|\Omega| = (b-a+1) < \infty$,
$$\mathcal{A} = \mathcal{P}(\Omega)$$
and $\mathbb{P}[\{\omega\}] = \dfrac{1}{|\Omega|} = \dfrac{1}{b-a+1}$, $\forall \omega \in \Omega = \{1, \dots, b-a+1\}$

Define a r.v. $X: \Omega \to \mathbb{R}$ as
$$X(\omega) = \omega + a - 1, \quad \forall \omega \in \Omega$$
(so that, $X(\omega_1 = 1) = a$, $X(\omega_2 = 2) = a+1, \dots, X(\omega_{b-a+1} = b-a+1) = b$

Then $\mathbb{P}[X = x] = \mathbb{P}[\omega \in \Omega : X(\omega) = x] = \mathbb{P}[\{\omega\}]$, where $\omega \in \Omega$,
$\forall x = \{a, \dots, b\}$

(10) The experiment $\mathcal{E}$ is a combination of 2 independent Random experiments, i.e., Alice picking a subset ($\mathcal{E}_1$) and Bob picking a subset ($\mathcal{E}_2$).

$$\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2$$

(10 a) $\mathcal{E}_1 : (\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ where

$\Omega_1 \Rightarrow$ Set of all possible subsets of $\{1, \ldots, N\}$ (contains 2^N elements)

$\mathcal{F}_1 \Rightarrow \mathcal{P}(\Omega_1)$ (Power set of Ω_1)

$\mathbb{P}_1 \Rightarrow \mathbb{P}_1[\{\omega\}] = \frac{1}{2^N}$, $\omega \in \Omega_1$ (each set is equally likely)

$\mathcal{E}_2 : (\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$

$\Omega_2 =$ Set of all possible subsets of $\{N, N+1, \ldots, 2N-1\}$

$\mathcal{F}_2 = \mathcal{P}(\Omega_2)$

$\mathbb{P}_2[\{\omega\}] = \frac{1}{2^N}$, $\omega \in \Omega_2$

For the compound experiment,

$\mathcal{E} \Rightarrow (\Omega, \mathcal{F}, \mathbb{P})$

$$\Omega \equiv \Omega_1 \times \Omega_2 = \{(\omega_A, \omega_B) : \omega_A \in \Omega_1, \omega_B \in \Omega_2\} //$$

$$\mathcal{F} \equiv \mathcal{F}_1 \times \mathcal{F}_2 //$$

$$\mathbb{P}[A] = \mathbb{P}[A_1 \times A_2] \quad [A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2]$$
$$\equiv \mathbb{P}_1[A_1] \cdot \mathbb{P}_2[A_2] //$$

(b) If the event $N \in A_2$ is A^*,

$$\mathbb{P}[A^*] = \mathbb{P}[\Omega_1 \times \{\omega_2 \in \Omega_2 : N \in \omega_2\}]$$
$$= \mathbb{P}_1[\Omega_1] \cdot \mathbb{P}_2[\{A_2 \in \mathcal{F}_2 : N \in A_2\}]$$
$$= 1 \times \frac{2^{N-1}}{2^N} = 1/2 //$$

(This is obtained by counting, There are 2^{N-1} elements in Ω_2 that contains N)

(c) $\mathbb{P}[A_1 \text{ and } A_2 \text{ do not intersect}]$

$= \mathbb{P}[\{\omega = (\omega_1, \omega_2) \in \Omega : N \notin \omega_1 , \text{ or } N \notin \omega_2\}]$

$= 1 - \mathbb{P}[\{(\omega_1, \omega_2) \in \Omega : N \in \omega_1 , N \in \omega_2\}]$

$= 1 - \mathbb{P}_1[\{\omega_1 \in \Omega_1 : N \in \omega_1\}] . \mathbb{P}_2[\{\omega_2 \in \Omega_2 : N \in \omega_2\}]$

$= 1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ (From previous calculation)

$= \frac{3}{4}$ //