ENEE 324-01$^{*}$ / SPRING 2018

HOMEWORK # 7

Problem 1 – 6 $\Rightarrow$

check the solution manual.

Problem 7 :

$X : \Omega \to \mathbb{R}$, $Y : \Omega \to \mathbb{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ discrete with

$S_X = \{0, 1, \ldots, n\}$ and $S_Y = \mathbb{N}$.

$$\mathbb{P}[X = x, Y = y] = c \binom{n}{x} (1-p)^x p^{n-x} p^y, \quad x = 0, 1, \ldots, n, \quad y = 0, 1, \ldots$$

(7.a) $$\sum_{x \in S_X} \sum_{y \in S_Y} \mathbb{P}[X = x, Y = y] = 1$$

$$\sum_{x=0}^{n} \sum_{y=0}^{\infty} c \binom{n}{x} (1-p)^x p^{n-x} p^y = 1$$

$$c \cdot \sum_{y=0}^{\infty} p^y \left[ \sum_{x=0}^{n} \binom{n}{x} (1-p)^x p^{n-x} \right] = 1$$

$$c \left[ \frac{1}{1-p} \right] = 1$$

$$c = 1 - p \quad //$$

(7.b) $$\mathbb{P}[X = x] = \sum_{y \in S_Y} \mathbb{P}[X = x, Y = y]$$

$$= \sum_{y=0}^{\infty} (1-p) \binom{n}{x} (1-p)^x p^{n-x} p^y$$

$$= (1-p) \binom{n}{x} (1-p)^x p^{n-x} \sum_{y=0}^{\infty} p^y$$

$$= (1-p) \binom{n}{x} (1-p)^x p^{n-x} \left( \frac{1}{1-p} \right)$$

$$= \binom{n}{x} (1-p)^x p^{n-x} \quad //$$

$\Rightarrow$ a Binomial distribution.

$$\mathbb{P}[Y=y] = \sum_{x\in S_X} \mathbb{P}[X=x,\ Y=y]$$

$$= \sum_{x=0}^{n} (p-1)\binom{n}{x}(1-p)^{x}(p)^{n-x}\ p^{y}$$

$$= (p-1)\,p^{y}\ \Big[\underbrace{\sum_{x=0}^{n}\binom{n}{x}(1-p)^{x}(p)^{n-x}}_{1}\Big]$$

$$= (p-1)\,p^{y} \quad // \qquad (\Rightarrow \text{a Geometric RV})$$

(7.c) $\Omega \longrightarrow \mathbb{R}^2$, $(X(\omega), Y(\omega))$ is indeed discreet. $S_X = \{0,1,\dots,n\}$ and $S_Y = \mathbb{N}$. consider $Z(\omega) = (X(\omega), Y(\omega))$ and $S_Z = S_X \times \mathbb{N}$.

$$\mathbb{P}[Z \in S_Z]$$
$$= \mathbb{P}[(X,Y) \in (S_X \times \mathbb{N})]$$
$$= \mathbb{P}[X \in S_X,\ Y \in \mathbb{N}]$$
$$= 1 \ // \qquad (\because \mathbb{P}(X \in S_X) = \mathbb{P}(Y \in \mathbb{N}) = 1)$$

Hence, $Z(\omega)$ is a discrete random variable with support $S_Z$ $(\{0,1,\dots,n\} \times \mathbb{N}]$.

Problem 8

$$\mathbb{P}(X = k) = (1-p)^{k-1}p \quad , \quad k = 1, 2, \ldots$$

$$\mathbb{E}(X) = \sum_{x \in S_X} x \, \mathbb{P}(X = x)$$

$$= \sum_{k=1}^{\infty} k \, \mathbb{P}(X = k)$$

$$= \sum_{k=1}^{\infty} k(1-p)^{k-1} p$$

$$= p\left[\sum_{k=1}^{\infty} k(1-p)^{k-1}\right] \text{ ——— } (1)$$

$$\sum_{k=0}^{\infty} (1-p)^k = \frac{1}{1-(1-p)} = \frac{1}{p}$$

By differentiating both sides once,

$$\frac{d}{dp}\left(\sum_{k=0}^{\infty} (1-p)^k\right) = \frac{-1}{p^2}$$

$$-\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{-1}{p^2} \quad \text{(Using the fact given in the problem)}$$

$$\sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2} \text{ ——— } (2)$$

$\therefore$ From (1);

$$\mathbb{E}(X) = p\left(\frac{1}{p^2}\right)$$

$$= 1/p \;//$$

Similarly,

$$\mathbb{E}(X^2) = \sum_{k=1}^{\infty} k^2(1-p)^{k-1} p$$

$$= p\left[\sum_{k=1}^{\infty} k^2(1-p)^{k-1}\right] \text{ ——— } (3)$$

Taking the derivatives of ②;

$$\frac{d}{dp}\left(\sum_{k=1}^{\infty} k(1-p)^{k-1}\right) = \frac{d}{dp}\left(\frac{1}{p^2}\right)$$

$$-\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} = \frac{-2p}{p^4}$$

$$\sum_{k=2}^{\infty} k^2(1-p)^{k-2} - \sum_{k=2}^{\infty} k(1-p)^{k-2} = \frac{2}{p^3}$$

$$\sum_{k=2}^{\infty} k^2(1-p)^{k-1} - \sum_{k=2}^{\infty} k(1-p)^{k-1} = \frac{2(1-p)}{p^3} \quad \left(\text{Multiplying by } 1-p\right)$$

$$\sum_{k=1}^{\infty} k^2(1-p)^{k-1} - \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{2(1-p)}{p^3} \quad \left(\text{The terms at } k=1 \text{ cancel each other}\right)$$

$$\therefore \sum_{k=1}^{\infty} k^2(1-p)^{k-1} = \frac{2(1-p)}{p^3} + \frac{1}{p^2} \quad (\text{from ②})$$

$$= \frac{2-p}{p^3}$$

Hence, from ③;

$$E(X^2) = \frac{2-p}{p^2} \quad //$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$= \frac{1-p}{p^2} \quad //$$

10.a) $X_i : \Omega \to \mathbb{R}$, $i = 1, \dots, N$

$\mathbb{P}[X_i \in S_i] = 1$, $i = 1, \dots, N$

$\Rightarrow \mathbb{P}[\{\omega \in \Omega : X_i(\omega) \in S_i\}] = 1$, $i = 1, \dots, N$.

$\Rightarrow \mathbb{P}[\{\omega \in \Omega : X_i(\omega) \in S_i\}^c] = 0$, $i = 1, \dots, N$. ——— (i)

Now, $\mathbb{P}[X_1 \in S_1, X_2 \in S_2, \dots, X_N \in S_N]^c$

$$= \mathbb{P}[\{\omega \in \Omega : X_1(\omega) \in S_1\} \cap \dots \cap \{\omega \in \Omega : X_N(\omega) \in S_N\}]^c$$

$$= \mathbb{P}[\bigcap_{i=1}^{N} \{\omega \in \Omega : X_i(\omega) \in S_i\}]^c$$

$$= \mathbb{P}[\bigcup_{i=1}^{N} \{\omega \in \Omega : X_i(\omega) \in S_i\}^c] \quad \text{[ De-Morgan's law ]}$$

$$\leq \sum_{i=1}^{N} \mathbb{P}[\{\omega \in \Omega : X_i(\omega) \in S_i\}^c] \quad \text{[ Boole's inequality ]}$$

$$\leq \sum_{i=1}^{N} 0 \quad \text{[ from (i) ]}$$

$$= 0$$

So, $\mathbb{P}[X_1 \in S_1, X_2 \in S_2, \dots, X_N \in S_N]^c \leq 0$

$\Rightarrow \mathbb{P}[X_1 \in S_1, \dots, X_N \in S_N]^c = 0$ [ cannot be $-$ve ]

$\Rightarrow \mathbb{P}[X_1 \in S_1, \dots, X_N \in S_N] = 1$ (Proved)

10.b) $g : \mathbb{R}^N \to \mathbb{R}$

$Z : \Omega \to \mathbb{R}^N$

$Z = g(X_1, X_2, \dots, X_N)$

To show that $Z$ is a discrete rv, we have to show $\exists$ a countable set $S_Z \subseteq \mathbb{R}^N$ s.t. $\mathbb{P}[Z \in S_Z] = 1$.

Consider $S_Z \triangleq \{g(x_1, \dots, x_N) : (x_1, \dots, x_N) \in (S_1 \times \dots \times S_N)\}$

Since $S_i$, $i=1,\ldots,N$ are all countable subsets of $\mathbb{R}$, (given)

$S_1 \times \cdots \times S_N$ is also countable and a subset of $\mathbb{R}^N$

$\Rightarrow \{(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N)\}$ is countable and a subset of $\mathbb{R}^N$

$\Rightarrow \{g(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N)\}$ is countable and a subset of $\mathbb{R}^N$

[$\because$ For each $(x_1,\ldots,x_N)$, $g$ maps $(x_1,\ldots,x_N)$ to only one element in its range]

$\Rightarrow S_Z$ is countable and a subset of $\mathbb{R}^N$

Now, $\mathbb{P}[Z \in S_Z] = \mathbb{P}[g(X_1,\ldots,X_N) \in S_Z]$

$= \mathbb{P}[g(X_1,\ldots,X_N) \in \{g(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N)\}]$

$= \mathbb{P}[(X_1,\ldots,X_N) \in \{(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N)\}]$

$= \mathbb{P}[(X_1,\ldots,X_N) \in (S_1 \times \cdots \times S_N)]$

$= \mathbb{P}[X_1 \in S_1, \ldots, X_N \in S_N]$

$= 1$ [from (a)]

Hence, $Z$ is a discrete r.v. (Proved).

Its pmf $p_Z(z) = \mathbb{P}[Z = z]$, $z \in S_Z$

$= \mathbb{P}[g(X_1,\ldots,X_N) = z]$, $z \in S_Z$

$= \mathbb{P}[g(X_1,\ldots,X_N) = g(x_1,\ldots,x_N)]$, [$\because (x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N)$ and $g(x_1,\ldots,x_N) = z$]

$= \mathbb{P}[(X_1,\ldots,X_N) \in \{(x_1,\ldots,x_N) : (x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N) \text{ and } g(x_1,\ldots,x_N) = z\}]$

$$= \sum_{\substack{(x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N) \\ \text{and } g(x_1,\ldots,x_N) = z}} \mathbb{P}[(X_1, X_2,\ldots,X_N) = (x_1,\ldots,x_n)]$$

$$= \sum_{\substack{(x_1,\ldots,x_N) \in (S_1 \times \cdots \times S_N) \\ : g(x_1,\ldots,x_N) = z}} \mathbb{P}[X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N]$$

$$\therefore p_Z(z) = \sum_{\substack{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N) \\ : g(x_1,\dots,x_N)=z}} p_{X_1,\dots,X_N}(x_1,\dots,x_N), \quad \forall z \in S_z$$

where $\{p_{X_1,\dots,X_N}(x_1,\dots,x_N), (x_1,\dots,x_N)\in(S_1\times\cdots\times S_N)\}$ is their joint pmf.

10.c) For a discrete r.v. $Z$,

$$\mathbb{E}[Z] = \sum_{z\in S_z} z\,\mathbb{P}[Z=z]$$

$$= \sum_{z\in S_z} z \cdot \left[\sum_{\substack{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N) \\ : g(x_1,\dots,x_N)=z}} p_{X_1,\dots,X_N}(x_1,\dots,x_N)\right] \quad [\text{from (b)}]$$

$$= \sum_{z\in S_z}\left[\sum_{\substack{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N) \\ : g(x_1,\dots,x_N)=z}} z\,p_{X_1,\dots,X_N}(x_1,\dots,x_N)\right]$$

$$= \sum_{z\in S_z}\left[\sum_{\substack{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N) \\ : g(x_1,\dots,x_N)=z}} g(x_1,\dots,x_N)\,p_{X_1,\dots,X_N}(x_1,\dots,x_N)\right]$$

[$\because$ Inside the second (inner) sum, $g(x_1,\dots,x_N)=z$]

$$= \sum_{z\in S_z}\left[\sum_{\substack{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N) \\ : g(x_1,\dots,x_N)=z}} g(x_1,\dots,x_N)\cdot p_{X_1,\dots,X_N}(x_1,\dots,x_N)\right]$$

$$= \sum_{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N)} g(x_1,\dots,x_N)\cdot p_{X_1,\dots,X_N}(x_1,\dots,x_N),$$

assuming that $\sum_{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N)} |g(x_1,\dots,x_N)|\,p_{X_1,\dots,X_N}(x_1,\dots,x_N)$ exists and finite.

(Proved)

9. a) $X_i : \Omega \to \mathbb{R}$, $i=1,\dots,N$ are discrete r.v. with
$\mathbb{P}[X_i \in S_i] = 1$, $i=1,\dots,N$
and $S_i$ countable subset of $\mathbb{R}$, $i=1,\dots,N$.

$Z : \Omega \to \mathbb{R}$

$$Z = \prod_{i=1}^{N} X_i$$

To show that $Z$ is discrete r.v, we have to show $\exists$ a countable $S_Z \subset \mathbb{R}$ s.t. $\mathbb{P}[Z \in S_Z] = 1$.

Define $S_Z \triangleq \{x_1 x_2 \dots x_N : (x_1,\dots,x_N) \in (S_1 \times \cdots \times S_N)\}$

Given $S_i$, $i=1,\dots,N$ are countable subsets of $\mathbb{R}$,

$S_1 \times \cdots \times S_N$ is a countable subset of $\mathbb{R}^N$

$\iff \{(x_1,\dots,x_N) : (x_1,\dots,x_N) \in (S_1 \times \cdots \times S_N)\}$ is countable subset of $\mathbb{R}^N$

$\Rightarrow \{x_1 x_2 \dots x_N : (x_1,\dots,x_N) \in (S_1 \times \cdots \times S_N)\}$ is countable subset of $\mathbb{R}$.

$\iff S_Z$ is countable subset of $\mathbb{R}$.

Now, $\mathbb{P}[Z \in S_Z] = \mathbb{P}[X_1 \cdots X_N \in S_Z]$

$= \mathbb{P}[X_1 \cdots X_N \in \{x_1 x_2 \dots x_N : (x_1,\dots,x_N) \in (S_1 \times \cdots \times S_N)\}]$

$= \mathbb{P}[(X_1,\dots,X_N) \in \{(x_1,\dots,x_N) : (x_1,\dots,x_N) \in (S_1 \times \cdots \times S_N)\}]$

$= 1$ [As done in Problem (10)]

$\therefore Z$ is discrete with support $S_Z = \{x_1 x_2 \cdots x_N : (x_1,\dots,x_N) \in (S_1 \times \cdots \times S_N)\}$.

9.b) Proceeding in same way as 10.(b),

the pmf of $Z$ is

$$p_Z(z) = \sum_{\substack{(x_1,\dots,x_N)\in(S_1\times\cdots\times S_N) \\ : z = x_1 x_2 \dots x_n}} p_{X_1,\dots,X_N}(x_1,\dots,x_N) \quad , \quad \forall\ z \in S_z$$

where $\{p_{X_1,\dots,X_N}(x_1,\dots,x_N) ; (x_1,\dots,x_N)\in(S_1\times\cdots\times S_N)\}$ is the joint pmf of the rvs $X_1,\dots,X_N$.

Hence, we need the knowledge of the joint pmfs of $X_1,\dots,X_N$ to compute pmf of $Z$. The individual pmfs of $X_1,\dots,X_N$ would not be sufficient in general.