ENEE 324 - 01* / SPRING 2018

HOMEWORK #8

SOLUTIONS

Problem 1, 2 ⇒ Look up the solution manual.

(3) $\mathbb{P}(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}, \quad x \in \mathbb{N}$

$\mathbb{P}[Y=y] = \mathbb{P}[X=y \mid X>0], \quad y=0,1,\ldots$

$\mathbb{P}[Y=0] = \mathbb{P}[X=0 \mid X>0] = 0$

$\mathbb{P}[Y=y] = \dfrac{\mathbb{P}[X=y, X>0]}{\mathbb{P}[X>0]}, \quad y=1,2,\ldots$

$= \dfrac{e^{-\lambda}\lambda^{y}}{y!} \times \dfrac{1}{(1-e^{-\lambda})}$

$$\therefore \mathbb{E}(Y) = \sum_{y=1}^{\infty} \frac{y e^{-\lambda}\lambda^{y}}{y!} \times \frac{1}{(1-e^{-\lambda})}$$

$$= \frac{e^{-\lambda}\lambda}{(1-e^{-\lambda})}\Bigg(\underbrace{\sum_{y=1}^{\infty}\frac{\lambda^{y-1}}{(y-1)!}}_{e^{\lambda}}\Bigg)$$

$$= \frac{\lambda}{1-e^{-\lambda}} > \lambda = \mathbb{E}(X) \qquad \left(\because 1-e^{-\lambda} < 1\right)$$

$\mathbb{E}(Y) > \mathbb{E}(X)$ is not a suprising result. The random variable Y has been formed by excluding the smallest possible value of X (which is zero) and increasing the probabilities of the other values of X. Hence, naturally, Y has a greater expectation than X.

(3.b) $Var(Y) = E\left[(Y - E(Y))^2\right]$

$= E(Y^2) - (E(Y))^2$

$$E(Y^2) = \sum_{y=1}^{\infty} y^2 \frac{e^{-\lambda}\lambda^y}{y!} \times \frac{1}{(1-e^{-\lambda})}$$

$$= \frac{e^{-\lambda}}{(1-e^{-\lambda})} \sum_{y=1}^{\infty} \frac{(y-1+1)\,\lambda^y}{(y-1)!}$$

$$= \frac{1}{(1-e^{-\lambda})}\left[\lambda \sum_{y=1}^{\infty} \frac{(y-1)\lambda^{y-1}e^{-\lambda}}{(y-1)!} + \lambda \sum_{y=1}^{\infty} \frac{e^{-\lambda}\lambda^{y-1}}{(y-1)!}\right]$$

$$= \frac{1}{1-e^{-\lambda}}\left(\lambda \sum_{k=0}^{\infty} \frac{k\,\lambda^k e^{-\lambda}}{k!} + \lambda \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}\right]$$

$$= \frac{1}{1-e^{-\lambda}}\left[\lambda \cdot \lambda + \lambda\right]$$

$$\therefore Var(Y) = \frac{1}{(1-e^{-\lambda})}(\lambda^2+\lambda) - \frac{\lambda^2}{(1-e^{-\lambda})^2}$$

$$= \frac{(1-e^{-\lambda})(\lambda^2+\lambda) - \lambda^2}{(1-e^{-\lambda})^2}$$

$$= \frac{\lambda - \lambda(\lambda+1)e^{-\lambda}}{(1-e^{-\lambda})^2} \quad //$$

(3.c) $$\frac{Var(Y)}{Var(X)} = \frac{\lambda - \lambda(\lambda+1)e^{-\lambda}}{(1-e^{-\lambda})^2} \times \frac{1}{\lambda}$$

$$= \frac{1-(1+\lambda)e^{-\lambda}}{(1-e^{-\lambda})^2}$$

Consider the function

$$g(\lambda) = Var(Y) - Var(X)$$

$$= 1-(1+\lambda)e^{-\lambda} - \left[1-e^{-\lambda}\right]^2$$

$$= -(1+\lambda)e^{-\lambda} + 2e^{-\lambda} - e^{-2\lambda}$$

$$= e^{-\lambda}\left(1-\lambda-e^{-\lambda}\right)$$

Consider $f(\lambda) = 1-\lambda-e^{-\lambda}$

$f(0) = 0$

$$f'(\lambda) = -1 + e^{-\lambda}$$

$$\Rightarrow f'(0) = 0$$

$$f''(\lambda) = -e^{-\lambda}$$

$$f''(0) = -1 < 0$$

Hence, $\lambda = 0$ is a maximum of $f(\lambda)$ (In fact, this is the only stationary point of $f(\lambda)$)

$$\therefore f(\lambda) < f(0), \quad \lambda > 0$$

$$f(\lambda) < 0, \quad \lambda > 0$$

$$\therefore g(\lambda) = e^{-\lambda} f(\lambda) < 0, \quad \lambda > 0 \quad (\because e^{-\lambda} \geq 0)$$

$$\Rightarrow Var(Y) - Var(X) < 0, \quad \lambda > 0$$

$$Var(Y) < Var(X), \quad \lambda > 0$$

Hence, $Var(X)$ is greater than $Var(Y)$ for $\lambda > 0$.

(4) $\mathbb{P}[X_k = x] = a_k(1-a_k)^x \qquad x = 0, 1, \ldots$

$X_1 \perp\!\!\!\perp X_2 \qquad k = 1, 2 \quad , \; 0 < a_1, a_2 < 1$

(4.a) $\mathbb{P}[X_1 + X_2 = y] \qquad y = 0, 1, \ldots$

$$= \sum_{r=0}^{y} \mathbb{P}[X_1 = r, X_2 = y - r] \qquad \text{(The events are disjoint)}$$

$$= \sum_{r=0}^{y} \mathbb{P}[X_1 = r]\, \mathbb{P}[X_2 = y - r] \qquad (\because X_1 \perp\!\!\!\perp X_2)$$

$$= \sum_{r=0}^{y} a_1(1-a_1)^r \cdot a_2(1-a_2)^{y-r} \qquad \text{— (1)}$$

$$= a_1 a_2 \left[\sum_{r=0}^{y} \left(\frac{1-a_1}{1-a_2}\right)^r\right] (1-a_2)^y$$

$$= a_1 a_2 \frac{\left[1 - \left(\frac{1-a_1}{1-a_2}\right)^{y+1}\right]}{1 - \left(\frac{1-a_1}{1-a_2}\right)} (1-a_2)^y$$

$$= \frac{a_1 a_2}{a_1 - a_2} \left[(1-a_2)^{y+1} - (1-a_1)^{y+1}\right] \qquad y = 0, 1, 2 \ldots$$

//

(4.b) $\mathbb{P}[X_1 = x \mid X_1 + X_2 = y] \qquad x = 0, \ldots, y$

$$= \frac{\mathbb{P}[X_1 = x, X_1 + X_2 = y]}{\mathbb{P}(X_1 + X_2 = y)}$$

$$= \frac{\mathbb{P}(X_1 = x, X_2 = y - x)}{\mathbb{P}(X_1 + X_2 = y)}$$

$$= \frac{\mathbb{P}(X_1 = x) \cdot \mathbb{P}(X_2 = y - x)}{\mathbb{P}(X_1 + X_2 = y)} \qquad (X_1 \perp\!\!\!\perp X_2)$$

$$= \frac{a(1-a)^x \cdot a(1-a)^{y-x}}{a^2 (1-a)^y (y+1)} \qquad \text{[denominator obtained by substituting } a_1 = a_2 = a \text{ in (1)]}$$

$$= \frac{1}{y+1}, \qquad x = 0, 1, \ldots, y$$

This is a uniform distribution over the set $\{0, 1, 2, \ldots, y\}$.

(4 c) $\mathbb{E}[X_1 \mid X_1 + X_2 = y]$ $\quad y = 0, 1, \ldots$

$$= \sum_{x=0}^{y} x \, \mathbb{P}[X_1 = x \mid X_1 + X_2 = y]$$

$$= \sum_{x=0}^{y} x \times \frac{1}{(y+1)} \qquad (\text{from } (b))$$

$$= \frac{1}{(y+1)} \cdot \frac{y(y+1)}{2}$$

$$= \frac{y}{2} \,//$$

Problem 5,6 $\Rightarrow$ Look up the solution manual.

7) $X \sim Pois(\lambda)$
$Y \sim Pois(\mu)$
$X \perp\!\!\!\perp Y$

a) $\mathbb{P}[X+Y=z \mid X=x]$, $x, y = 0, 1, \ldots$

$= \mathbb{P}[x+Y=z \mid X=x]$

$= \mathbb{P}[Y=z-x \mid X=x]$

$= \mathbb{P}[Y=z-x]$ $[\because X \perp\!\!\!\perp Y]$

$$= \begin{cases} e^{-\mu} \cdot \dfrac{\mu^{z-x}}{(z-x)!} & , \text{if } z \geq x \\ 0 & , \text{if } z < x \end{cases} \quad ; x \in \{0,1,\ldots\}, y \in \{0,1,\ldots\}.$$

b) $\mathbb{P}[X=x \mid X+Y=z]$

$$= \frac{\mathbb{P}[X+Y=z \mid X=x] \cdot \mathbb{P}[X=x]}{\mathbb{P}[X+Y=z]}$$

The numerator is known from part (a).

The denominator $\mathbb{P}[X+Y=z] = \sum_{x=0}^{\infty} \mathbb{P}[X+Y=z \mid X=x] \cdot \mathbb{P}[X=x]$ $[\because \mathbb{P}[X=x]=0$ if $x<0]$

$= \sum_{x=0}^{z} e^{-\mu} \cdot \frac{\mu^{z-x}}{(z-x)!} \cdot e^{-\lambda} \cdot \frac{\lambda^x}{x!}$ $[\because \mathbb{P}[X+Y=z \mid X=x] = 0$ if $x>z]$ (if $z \geq 0$)

$= e^{-(\mu+\lambda)} \frac{\mu^z}{z!} \sum_{x=0}^{z} \lambda^x \cdot \mu^{-x} \cdot \frac{z!}{(z-x)!\,x!}$

$= e^{-(\mu+\lambda)} \cdot \frac{\cancel{\mu^z}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \cdot \mu^{z-x}$

$= e^{-(\mu+\lambda)} \cdot \frac{1}{z!} (\lambda+\mu)^z$

$= e^{-(\mu+\lambda)} \cdot \frac{(\lambda+\mu)^z}{z!}$, $z \geq 0$.

$\Rightarrow X+Y \sim Pois(\lambda+\mu)$

i.e., sum of two independent Poisson r.v. is Poisson with the parameter being equal to the sum of the former parameters.

$\therefore\ \mathbb{P}[X=x \mid X+Y=z]$

$$= \frac{e^{-\mu}.\mu^{z-x}}{(z-x)!} \times \frac{e^{-\lambda}.\lambda^{x}}{x!} \times \frac{1}{e^{-(\lambda+\mu)}.\frac{(\lambda+\mu)^{z}}{z!}}, \quad \text{if } z \geqslant x$$

$$= \frac{z!}{(z-x)!\,x!} \cdot \frac{e^{-\mu}.e^{-\lambda}}{e^{-(\lambda+\mu)}} \cdot \frac{\lambda^{x}.\mu^{z-x}}{(\lambda+\mu)^{z}}$$

$$= \begin{cases} \binom{z}{x}\left(\frac{\lambda}{\mu}\right)^{x}\left(1+\frac{\lambda}{\mu}\right)^{-z}, & \text{if } z \geqslant x \\ 0, & \text{if } z < x. \end{cases}$$

8) $\xi_1, \xi_2, \ldots, \xi_n$ are n r.v.

$\mathbb{P}[\xi_k \in \{0,1,\ldots,9\}] = 1, \forall k = 1,\ldots,n$

$X := 0.\xi_1\xi_2\ldots\xi_n$ (decimal expansion) [e.g., $x = 0.\underset{\uparrow \xi_1}{2}321\ldots\underset{\uparrow \xi_n}{7}$ etc]

So, $X = 10^{-n}\xi_n + 10^{-(n-1)}\xi_{n-1} + 10^{-(n-2)}\xi_{n-2} + \cdots + 10^{-1}\xi_1 + 0$

$$= \sum_{k=1}^{n} 10^{-k}\xi_k$$

a) $\mathbb{E}[X] = \mathbb{E}\left[\sum_{k=1}^{n} 10^{-k}\xi_k\right]$

$$= \sum_{k=1}^{n} 10^{-k}\,\mathbb{E}[\xi_k] \quad \text{[by Linearity of expectation operation]}$$

If $\mathbb{E}[\xi_k] = m, \forall k = 1,\ldots,n$,

$$\mathbb{E}[X] = \sum_{k=1}^{n} 10^{-k}.m$$

$$= m\sum_{k=1}^{n} 10^{-k}$$

$$= m \times (10^{-1}).\frac{1-(10^{-1})^{n}}{1-(10^{-1})} \quad \text{[sum of a GP with } r = 10^{-1}\text{]}$$

$$= m \times 0.1 \times \frac{1-(0.1)^{n}}{1-0.1} = \frac{m}{9}\left(1-(0.1)^{n}\right).$$

b) ξ_k is uniform on $\{0,1,\ldots,9\}$, $\forall k=1,\ldots,n$

$$\mathbb{P}[\xi_k = x] = \frac{1}{10}, \quad x=0,\ldots,9, \quad \forall k=1,\ldots,n$$

$$\therefore \mathbb{E}[\xi_k] = \sum_{x=0}^{9} x \cdot \mathbb{P}[\xi_k = x], \quad k=1,\ldots,n$$

$$= \sum_{x=0}^{9} x \cdot \frac{1}{10}$$

$$= \frac{1}{10} \times \frac{9\times 10}{2}$$

$$= 9/2.$$

$$\therefore \mathbb{E}[X] = \frac{1}{9}\left(\frac{9}{2}\right)\cdot\left(1-(0.1)^n\right) \quad [\because m = 9/2 \text{ now}]$$

$$= \frac{1}{2}\left(1-(0.1)^n\right).$$

When n becomes very large,

$$\lim_{n\to\infty} \mathbb{E}[X] = \lim_{n\to\infty} \frac{1}{2}\left(1-(0.1)^n\right)$$

$$= \frac{1}{2}$$

(As $n\to\infty$, X becomes uniformly distributed on $[0,1)$. In that case, $\mathbb{E}[X]$ is expected to be 0.5.)

Problem 9, 10 $\Rightarrow$ Look up the solution manual.