
ENGINEERING PROBABILITY

HOMEWORK # 9:
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Please work out the **ten** (10) problems stated below – BT refers to the text: D.P. Bertsekas and J.N. Tsitsiklis, Introduction to Probability (Second Edition), Athena Scientific (2008). Problem **1.55** (BT) refers to Problem 55 for Chapter 1 of BT (to be found at the end of Chapter 1). **Show** work and **explain** reasoning.

1.

Consider a discrete rv $X : \Omega \to \mathbb{R}$ with support $S = \mathbb{N}$ such that $\mathbb{P}[X = x] > 0$ for each $x = 0, 1, \ldots$.

1.a. For each $t = 0, 1, \ldots$, compute the conditional probabilities

$$\mathbb{P}\left[(X - t)^+ = x | X \geq t\right], \quad x = 0, 1, \ldots$$

1.b. Is it possible to find a pmf for the rv X (with support $S = \mathbb{N}$) so that we simultaneously have

$$\mathbb{P}\left[(X - t)^+ = x | X \geq t\right] = \mathbb{P}[X = x], \quad \begin{array}{c} x = 0, 1, \ldots \\ t = 0, 1, \ldots \end{array}$$

2.

We start with two independent rvs $X, Y : \Omega \to \mathbb{R}$ which are discrete and hose supports are contained in $\mathbb{N}$. If X is a Binomial rv $\text{Bin}(n; a)$ and Y is a Binomial rv $\text{Bin}(m; a)$ for arbitrary positive integers n and m with $0 < a < 1$,

2.a. Find the pmf of the rv $X + Y$.

2.a. Evaluate the conditional expectation $\mathbb{E}[X | X + Y = z]$ for each $z = 0, 1, \ldots, n + m$.

3.

The rvs $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ are mutually independent *geometric* rvs on $\mathbb{N}$ (not on $\mathbb{N}_0$), say

$$\mathbb{P}[X_k = x] = a_k (1 - a_k)^x, \quad \begin{array}{c} x = 0, 1, \ldots \\ k = 1, \ldots, n \end{array}$$

with arbitrary parameters $0 < a_1, \ldots, a_n < 1$.

3.a. Compute the probability

$$\mathbb{P}\left[X_1 = X_2 = \ldots = X_n\right].$$

3.b. If $a_1 = \ldots = a_n \equiv a$, what happens to $\mathbb{P}\left[X_1 = X_2 = \ldots = X_n\right]$ when n becomes large? Does it match your intuition?

4.

Consider a discrete rv $X : \Omega \to \mathbb{R}$ with support contained in $\mathbb{N}$.

4.a. Show that its expectation $\mathbb{E}\left[X\right]$ can also be computed by using the expression

$$\mathbb{E}\left[X\right] = \sum_{x=0}^{\infty} \mathbb{P}\left[X > n\right].$$

4.b. Use Part **a** to evaluate $\mathbb{E}\left[X\right]$ when X has a *geometric* pmf on $\mathbb{N}$ – The calculations are a lot simpler than the ones carried out using the direct approach!

5.

Without calculations explain why the variance $\mathrm{Var}\left[X\right]$ of a Binomial rv $\mathrm{Bin}(n;p)$ is given by $np(1-p)$ with positive integer n and $0 < p < 1$.

6.

With rvs $X_1, \ldots, X_n : \Omega \to \mathbb{R}$, we associate the rvs $X^\star, X_\star : \Omega \to \mathbb{R}$ given by

$$X^\star \equiv \max\left(X_1, \ldots, X_n\right) \quad \text{and} \quad X_\star \equiv \min\left(X_1, \ldots, X_n\right).$$

Assume that the rvs $X_1, \ldots, X_n$ are discrete rvs which are independent and identically distributed with common pmf $\{p(x),\ x \in S\}$ supported on the countable set $S \subseteq \mathbb{R}$, i.e.,

$$\mathbb{P}\left[X_i = x\right] = p(x), \qquad \begin{array}{c} x \in S \\ i = 1, \ldots, n. \end{array}$$

6.a. Find the pmf of each of the discrete rvs $X^\star$ and $X_\star$.

6.b. Apply the results to Part **a** to the situation when the common pmf is the geometric pmf on $\mathbb{N}$ given by

$$p(x) = a(1-a)^x, \quad x = 0, 1, \ldots$$

with $0 < a < 1$. Do you notice anything interesting?

7.

In this problem we consider evaluating $\mathbb{E}\left[\frac{1}{1+X}\right]$ when the rv $X : \Omega \to \mathbb{R}$ is a discrete rv with support contained in $\mathbb{N}$. Do the calculations when

7.a. the rv X is a Binomial rv $\mathrm{Bin}(n;p)$ with positive integer n and $0 < p < 1$.

7.b. the rv X is a Poison rv $\mathrm{Poi}(\lambda)$ with $\lambda > 0$.

8.

Consider the following setting: The $n+1$ discrete rvs $X_1, \ldots, X_n, \nu : \Omega \to \mathbb{R}$ are mutually independent rvs. We shall assume that the rvs $X_1, \ldots, X_n$ are discrete rvs which are independent and identically distributed with common pmf $\{p(x),\ x \in S\}$ supported on the countable set $S \subseteq \mathbb{R}$, i.e.,

$$\mathbb{P}\left[X_i = x\right] = p(x), \qquad \begin{array}{c} x \in S \\ i = 1, \ldots, n. \end{array}$$

Moreover, the rv ν is a discrete rv with support $\{0, 1, \ldots, N\}$ for some positive integer N. We shall assume that the (common) expectation of the rvs $X_1, \ldots, , X_n$ is finite.

Show that that the expression

$$\mathbb{E}\left[\sum_{k=1}^{\nu} X_i\right] = \mathbb{E}\left[\nu\right]\mathbb{E}\left[X_1\right]$$

holds; this formula is known as Wald's identity.

9.

Let X and Y be two independent rvs which are uniformly distributed on the set of integers $\{0, \ldots, 9\}$, i.e.,

$$\mathbb{P}\left[X = x\right] = \mathbb{P}\left[Y = y\right] = \frac{1}{10}, \quad x, y = 0, \ldots, 9.$$

You are told that their sum $X + Y$ is of the form

$$X + Y = \xi \cdot 10 + \eta$$

where ξ and η are discrete rvs taking values in $\{0, \ldots, 9\}$, i.e.,

$$\mathbb{P}\left[\xi \in \{0, \ldots, 9\}\right] = \mathbb{P}\left[\eta \in \{0, \ldots, 9\}\right] = 1.$$

9.a. Explain how you would go about computing the probabilities

$$\mathbb{P}\left[\xi = x, \eta = y\right], \quad x, y = 0, \ldots, 9.$$

9.b. Compute the probabilities

$$\mathbb{P}\left[\xi = 0, \eta = 0\right],\ \mathbb{P}\left[\eta = 0\right] \text{ and } \mathbb{P}\left[\xi = 0\right].$$

What can you say about the independence of the rvs ξ and η?

10.

A point X is picked uniformly at random from the set of integers $\{0, \ldots, \nu\}$ for some positive integer ν which is itself selected uniformly at random from the set $\{1, \ldots, N\}$ for some positive integer N. Thus,

$$\mathbb{P}\left[\nu = k\right] = \frac{1}{N}, \quad k = 1, 2, \ldots, N.$$

10.a. Find the probabilities

$$\mathbb{P}\left[\nu = k | X < j\right], \qquad \begin{array}{c} k = 1, 2, \ldots, N \\ j = 1, \ldots, N \end{array}$$

10.b. Compute

$$\mathbb{E}\left[\nu = k | X < j\right], \quad j = 1, \ldots, N$$

10.c. Compute $\mathbb{E}\left[X\right]$.
