ENEE324-01* / SPRING 2018

HOMEWORK #9

SOLUTIONS.

1) $X : \Omega \to \mathbb{R}$

X discrete with support $S = \mathbb{N}$, i.e., $\mathbb{P}[X=x] > 0$ if $x = 0,1,2,\ldots$

a) Fix $t \in \{0,1,\ldots\}$ and $x \in \{0,1,\ldots\}$.

Then, $\mathbb{P}[(X-t)^+ = x \mid X \geq t]$

$$= \frac{\mathbb{P}[(X-t)^+ = x, X \geq t]}{\mathbb{P}[X \geq t]} \quad [\text{With } t \in \{0,1,\ldots\},\ \mathbb{P}[X=t] > 0 \Rightarrow \mathbb{P}[X \geq t] > 0]$$

$$= \frac{\mathbb{P}[\max\{0, X-t\} = x, X-t \geq 0]}{\mathbb{P}[X \geq t]}$$

$$= \frac{\mathbb{P}[\cancel{X-t = x, X-t \geq 0}]}{\mathbb{P}[X \geq t]}$$

$$= \frac{\mathbb{P}[X-t = x]}{\mathbb{P}[X \geq t]}$$

[The event $\{X-t = x\}$ is a subset of $\{X-t \geq 0\}$ when $x \in \{0,1,\ldots\}$. Now, $A \subseteq B \Rightarrow A \cap B = A$
$\Rightarrow \mathbb{P}[A \cap B] = \mathbb{P}[A]$
$\Rightarrow \mathbb{P}[\{X-t = x\} \cap \{X-t \geq 0\}] = \mathbb{P}[\{X-t = x\}]$]

$$= \frac{\mathbb{P}[X = t+x]}{\mathbb{P}[X \geq t]}.$$

$$\therefore\ \mathbb{P}[(X-t)^+ = x \mid X \geq t] = \frac{\mathbb{P}[X = t+x]}{\mathbb{P}[X \geq t]}, \quad \begin{array}{l} x = 0,1,\ldots \\ t = 0,1,\ldots \end{array}$$

b) We have to find/check if there exist a pmf of the rv X (with support $S = \mathbb{N}$) s.t.,

$$\mathbb{P}[(X-t)^+ = x \mid X \geq t] = \mathbb{P}[X = x], \quad \begin{array}{l} x = 0,1,\ldots \\ t = 0,1,\ldots \end{array} \quad \text{—— (i)}$$

What discrete rvs do you know which have support $S = \mathbb{N}$?

Let's check if $X \sim \text{Geom}(p)$ with support $S = \mathbb{N}$ satisfies (i). $[0 < p < 1]$

$$\mathbb{P}[(X-t)^+ = x \mid X \geq t] = \frac{\mathbb{P}[X = t+x]}{\mathbb{P}[X \geq t]}$$

$$= \frac{(1-p)^{t+x}\cdot p}{1-\mathbb{P}[X \le t-1]} \qquad [\because \mathbb{P}[X \ge t] = 1-\mathbb{P}[X<t] = 1-\mathbb{P}[X \le t-1] \text{ as } t \in \{0,1,\dots\}]$$

$$= \frac{(1-p)^{t+x}\cdot p}{1-\{1-(1-p)^{t-1+1}\}} \qquad [\because \text{CDF of geometric rv } X \text{ with support } \mathbb{N} \text{ is } \mathbb{P}[X \le k] = 1-(1-p)^{k+1},\ k=0,1,\dots]$$

$$= \frac{(1-p)^{t+x}\cdot p}{(1-p)^{t}}$$

$$= (1-p)^{x}\cdot p \qquad [\text{which is indeed the pmf of } X \sim Geom(p) \text{ with support } \mathbb{N}]$$

$$= \mathbb{P}[X=x], \quad x = 0,1,\dots, \quad t = 0,1,\dots$$

2) $X \sim Bin(n;a)$
$Y \sim Bin(m;a)$
$n, m \in \mathbb{N}, \quad 0<a<1$
$X \perp\!\!\!\perp Y$

a) X has support $S_x = \{0,\dots,n\}$
Y has support $S_y = \{0,\dots,m\}$
$\therefore$ Clearly, $(X+Y)$ is discrete
and $(X+Y)$ has support $\{0,\dots,m+n\}$.

Then, $\mathbb{P}[X+Y=z], \quad z = 0,1,\dots,m+n$

$$= \sum_{l \in S_y} \mathbb{P}[X+Y=z, Y=l]$$

$$= \sum_{l=0}^{m} \mathbb{P}[X=z-l, Y=l]$$

$$= \sum_{l=0}^{m} \mathbb{P}[X=z-l]\cdot\mathbb{P}[Y=l] \qquad [\because X \perp\!\!\!\perp Y]$$

$$= \sum_{l=0}^{m} \binom{n}{z-l} a^{z-l}(1-a)^{n-z+l} \cdot \binom{m}{l} a^{l}(1-a)^{m-l}$$

$$= a^{z}(1-a)^{n+m-z} \cdot \sum_{l=0}^{m} \binom{n}{z-l}\binom{m}{l}$$

Now, $(1+x)^n(1+x)^m = (1+x)^{n+m}$

$$\Rightarrow \sum_{p=0}^{n}\binom{n}{p}x^{p} \cdot \sum_{l=0}^{m}\binom{m}{l}x^{l} = \sum_{z=0}^{n+m}\binom{n+m}{z}x^{z}$$

$$\Rightarrow \sum_{l=0}^{m} \sum_{r=0}^{n} \binom{n}{r}\binom{m}{l} x^{r+l} = \sum_{z=0}^{n+m} \binom{n+m}{z} x^{z}$$

Comparing the coefficients of x^z on both sides,

$$\sum_{l=0}^{m} \binom{n}{z-l}\binom{m}{l} = \binom{n+m}{z} \quad \text{——— (i)}$$

$$\therefore \mathbb{P}[X+Y=z] = \binom{n+m}{z} a^{z}(1-a)^{n+m-z}, \quad z=0,1,\dots,n+m$$

$$\Rightarrow \quad X+Y \sim Bin(n+m; a)$$

b) $\mathbb{E}[X \mid X+Y=z], \quad z=0,1,\dots,n+m$

$$= \sum_{x\in S_X} x \cdot \mathbb{P}[\cancel{X=x} \mid X+Y=z]$$

$$= \sum_{x=0}^{n} x \cdot \frac{\mathbb{P}[X=x, X+Y=z]}{\mathbb{P}[X+Y=z]}$$ [$\because \mathbb{P}[X+Y=z]>0$ when $z\in\{0,1,\dots,n+m\}$ because $X+Y \sim Bin(n+m;a)$]

$$= \sum_{x=0}^{n} x \cdot \frac{\mathbb{P}[X=x, Y=z-x]}{\mathbb{P}[X+Y=z]}$$

$$= \sum_{x=0}^{n} x \cdot \frac{\mathbb{P}[X=x]\,\mathbb{P}[Y=z-x]}{\mathbb{P}[X+Y=z]}$$ [$\because X \perp\!\!\!\perp Y$]

$$= \sum_{x=0}^{n} x \cdot \frac{\binom{n}{x} a^{x}(1-a)^{n-x} \cdot \binom{m}{z-x} a^{z-x}(1-a)^{m-z+x}}{\binom{n+m}{z} a^{z}(1-a)^{n+m-z}}$$

$$= \sum_{x=0}^{n} x \cdot \binom{n}{x} \cdot \binom{m}{z-x} \cdot \frac{1}{\binom{n+m}{z}}$$

$$= \frac{1}{\binom{n+m}{z}} \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} \times \frac{m!}{(z-x)!(m-z+x)!}$$

$$= \frac{1}{\binom{n+m}{z}} \sum_{x=1}^{n} x \cdot \frac{n!}{x(n-x)!} \times \frac{m!}{(z-x)!(m-z+x)!}$$ [$\because \binom{n}{x}\binom{m}{z-x}$ is non-zero]

$$= \frac{1}{\binom{n+m}{z}} \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \cdot \frac{m!}{(z-x)!(m-z+x)!}$$

$$= \frac{n}{\binom{m+n}{z}} \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} \cdot \frac{m!}{(z-x)!(m-z+x)!}$$

$$= \frac{n}{\binom{m+n}{z}} \sum_{l=0}^{n-1} \frac{(n-1)!}{l!\,(n-l-1)!} \cdot \frac{m!}{(z-l-1)!\,(m-z+l+1)!}$$ [changing the index to $l = x-1$]

$$= \frac{n}{\binom{m+n}{z}} \sum_{l=0}^{n-1} \binom{n-1}{l}\binom{m}{z-1-l}$$

$$= \frac{n}{\binom{m+n}{z}} \cdot \binom{m+n-1}{z-1}$$ [from (i); with $(n-1)$ in place of m, m in place of n, $(z-1)$ in place of z in (i)]

$$= \frac{n}{\frac{(m+n)!}{z!\,(m+n-z)!}} \times \frac{(m+n-1)!}{(z-1)!\,(m+n-1-z+1)!}$$

$$= \frac{n.z!\,(m+n-z)!}{(m+n)!} \times \frac{(m+n-1)!}{(z-1)!\,(m+n-z)!}$$

$$= \frac{nz}{m+n}.$$

Similarly, $\mathbb{E}[Y \mid X+Y=z] = \frac{mz}{m+n}$, $z = 0,1,\ldots,m+n$

$\therefore \mathbb{E}[X+Y \mid X+Y=z] = \mathbb{E}[X \mid X+Y=z] + \mathbb{E}[Y \mid X+Y=z]$ [by linearity of $\mathbb{E}[\cdot]$]

$$= \frac{mz}{m+n} + \frac{nz}{m+n}$$

$= z.$, which it should be. (Just a cross-check!)

3> $X_1, X_2, \ldots, X_n : \Omega \to \mathbb{R}$

$\{X_1, \ldots, X_n\}$ are mutually independent geometric on $\mathbb{N}$

$\mathbb{P}[X_k = x] = a_k(1-a_k)^x$, $x = 0,1,\ldots$
$k = 1,\ldots,n$

with $0 < a_1, \ldots, a_n < 1$.

a> $\mathbb{P}[X_1 = X_2 = \cdots = X_n] = \sum_{x \in S} \mathbb{P}[X_1 = \cdots = X_n = x]$ [S being the common support $\mathbb{N}$ of all X_k's]

$$
\begin{aligned}
&= \sum_{x=0}^{\infty} \mathbb{P}[X_1 = x, X_2 = x, \ldots, X_n = x] \\
&= \sum_{x=0}^{\infty} \mathbb{P}[X_1 = x] \cdot \mathbb{P}[X_2 = x] \cdots \mathbb{P}[X_n = x] \quad [\text{ by mutual independence}] \\
&= \sum_{x=0}^{\infty} \left(\prod_{k=1}^{n} \mathbb{P}[X_k = x] \right) \\
&= \sum_{x=0}^{\infty} \left(\prod_{k=1}^{n} \left(a_k (1-a_k)^x \right) \right) \\
&= \sum_{x=0}^{\infty} \left(\prod_{k=1}^{n} a_k \right) \left(\prod_{k=1}^{n} \left[(1-a_k)^x \right] \right) \\
&= \prod_{k=1}^{n} a_k \cdot \sum_{x=0}^{\infty} \left(\prod_{k=1}^{n} \left[(1-a_k)^x \right] \right) \\
&= \prod_{k=1}^{n} a_k \cdot \sum_{x=0}^{\infty} \left(\left[\prod_{k=1}^{n} (1-a_k) \right]^x \right) \\
&= \prod_{k=1}^{n} a_k \cdot \sum_{x=0}^{\infty} r^x \quad \left[\text{With } r \triangleq \prod_{k=1}^{n} (1-a_k) \right]
\end{aligned}
$$

$$
\begin{aligned}
0 < a_k < 1 \ , \forall k = 1, \ldots, n \ &\Rightarrow 0 < (1-a_k) < 1 \ , \forall k = 1, \ldots, n \\
&\Rightarrow 0 < \prod_{k=1}^{n} (1-a_k) < 1 \\
&\Rightarrow 0 < r < 1
\end{aligned}
$$

$$
\begin{aligned}
\therefore \mathbb{P}[X_1 = \cdots = X_n] &= \prod_{k=1}^{n} a_k \cdot \frac{1}{1-r} \\
&= \frac{\prod_{k=1}^{n} a_k}{1 - \prod_{k=1}^{n} (1-a_k)} .
\end{aligned}
$$

3.b) If $a_1 = a_2 = \cdots = a_n = a$,

$$\mathbb{P}[X_1 = \cdots = X_n] = \frac{\prod_{k=1}^{n} a}{1 - \prod_{k=1}^{n}(1-a)}$$

$$= \frac{a^n}{1-(1-a)^n} \quad \text{——— (i)}$$

So, $\lim_{n\to\infty} \mathbb{P}[X_1 = \cdots = X_n] = \lim_{n\to\infty} \frac{a^n}{1-(1-a)^n}$

$$= \lim_{n\to\infty} \frac{1}{a^{-n}-(a^{-1}-1)^n}$$

$= 0$ [$\because \lim_{n\to\infty} a^{-n} \to \infty$ if $0<a<1$

and $a^{-1}-1 > 1 \Rightarrow (a^{-1}-a)^{n} \xrightarrow{n\to} \infty$,

but $a^{-n} = (a^{-1})^n$ grows faster with n

than $(a^{-1}-1)^n$ because $0<a<1$, i.e., a^{-1}:

Does it match your intuition?

Yes. Because,

Define $y_n \triangleq \mathbb{P}[X_1 = \cdots = X_n] = \frac{a^n}{1-(1-a)^n}$ [from (i)]

Clearly, $0 < y_n < 1$ (by axiom of probability) $\Rightarrow \{y_n\}_{n=1}^{\infty}$ is a bounded sequence

Also, $\{y_n\}_{n=1}^{\infty}$ is a monotonically decreasing sequence. [check from the expression in (i)]

Hence, $\{y_n\}_{n=1}^{\infty}$ converges to its lower bound zero.

By intuition, if you consider a finite number of identical rvs, ~~the~~ less likely they would be to take a common value when the number of those rvs is increased.

4) $X : \Omega \to \mathbb{R}$

X discrete with support $S_X = \mathbb{N} \subseteq \{0, 1, \dots\}$

a) $\mathbb{E}[X] = \sum_{x \in S_X} x \cdot \mathbb{P}[X = x]$

$= \sum_{x \in \mathbb{N}} x \cdot \mathbb{P}[X = x]$ [$\because \mathbb{P}[X=x] = 0$, if $x \in \mathbb{N} - S_X$, hence the sum does not change if those x's are also included in the sum]

$= \sum_{x \in \mathbb{N}} x \left(\mathbb{P}[X \geqslant x] - \mathbb{P}[X \geqslant x+1] \right)$

$= \left(\mathbb{P}[X \geqslant 1] - \mathbb{P}[X \geqslant 2]\right) + \left(2\,\mathbb{P}[X \geqslant 2] - 2\,\mathbb{P}[X \geqslant 3]\right) + \left(3\,\mathbb{P}[X \geqslant 3] - 3\,\mathbb{P}[X \geqslant 4]\right) + \dots \infty$

$= \mathbb{P}[X \geqslant 1] + \mathbb{P}[X \geqslant 2] + \mathbb{P}[X \geqslant 3] + \dots \infty$ [by telescopic sum]

$= \sum_{x=1}^{\infty} \mathbb{P}[X \geqslant x]$

Now, $\mathbb{P}[X > x] = \mathbb{P}[X \in (x, x+1)] + \mathbb{P}[X \geqslant x+1]$, $\forall x \in \mathbb{N}$.

$= \mathbb{P}[X \geqslant x+1]$ [$\because \mathbb{P}[X \in (x, x+1)] = 0$, as $(x, x+1) \not\subset S_X$ when $x \in \mathbb{N}$]

$\therefore \mathbb{E}[X] = \sum_{x=1}^{\infty} \mathbb{P}[X \geqslant x]$

$= \sum_{x=0}^{\infty} \mathbb{P}[X \geqslant x+1]$

$= \sum_{x=0}^{\infty} \mathbb{P}[X > x]$ (Proved)

b) $X \sim Geom(p)$ with support $S_X = \mathbb{N} = \{0, 1, 2, \dots\}$

$$\Rightarrow \mathbb{P}[X = x] = (1-p)^x p, \quad x = 0, 1, \dots$$
$$0 < p < 1.$$

$$\Rightarrow \mathbb{P}[X \leq x] = \sum_{k=0}^{x} \mathbb{P}[X = k], \quad x = 0, 1, \dots$$
$$= \sum_{k=0}^{x} (1-p)^k p$$
$$= p \cdot \frac{1-(1-p)^{x+1}}{1-(1-p)}$$
$$= \cancel{p} \cdot \frac{1-(1-p)^{x+1}}{\cancel{p}}$$
$$= 1-(1-p)^{x+1}$$, which is the known cdf of geometric rv on $\mathbb{N}$.

Then, $\mathbb{E}[X] = \sum_{x=0}^{\infty} \mathbb{P}[X > x]$ (from (a))
$$= \sum_{x=0}^{\infty} \left(1 - \mathbb{P}[X \leq x]\right)$$
$$= \sum_{x=0}^{\infty} \left(1 - 1 + (1-p)^{x+1}\right)$$
$$= \sum_{x=0}^{\infty} (1-p)^{x+1}$$
$$= \frac{(1-p)}{1-(1-p)}$$
$$= \frac{1-p}{p}.$$

5) Consider a Binomial rv $X \sim Bin(n; p)$, $n \in \mathbb{N}_0$, $0 < p < 1$.

Then, $X = \sum_{k=1}^{n} X_k$

s.t. $X_k \sim Bernoulli(p)$, $k = 1, \dots, n$

and $X_1, \dots, X_n$ are mutually independent. (By def^n of Binomial rv)

$$\therefore \operatorname{Var}[X] = \operatorname{Var}\left[\sum_{k=1}^{n} X_k\right]$$

$$= \sum_{k=1}^{n} \operatorname{Var}[X_k] + \sum_{k \neq l}\sum \operatorname{Cov}[X_k, X_l]$$

$$= \sum_{k=1}^{n} \operatorname{Var}[X_k] \quad [\because X_1, \dots, X_n \text{ are mutually independent}$$

$$\Rightarrow X_1, \dots, X_n \text{ are pairwise independent}$$

$$\Rightarrow X_k, X_l \text{ are uncorrelated}, \forall k, l \; k \neq l$$

$$\Rightarrow \operatorname{Cov}[X_k, X_l] = 0 \quad \forall k, l, k \neq l. \;]$$

$$= \sum_{k=1}^{n} p(1-p) \quad [X_k \sim \text{Bernoulli}(p) \Rightarrow \operatorname{Var}[X_k] = p(1-p), \forall k]$$

$$= np(1-p).$$

To find the variance of a Bernoulli r.v.:

$$X_k \sim \text{Bernoulli}(p)$$

$$\Rightarrow \mathbb{P}[X_k = x] = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases}$$

$$\therefore \mathbb{E}[X_k] = \sum_{x \in S_{X_k}} x \cdot \mathbb{P}[X_k = x] = 1 \cdot p + 0 \cdot (1-p) = p$$

$$\therefore \operatorname{Var}[X_k] = \mathbb{E}[X_k^2] - \mathbb{E}[X_k]^2 = 1^2 \cdot p + 0^2 \cdot (1-p) - p^2$$

$$= p - p^2$$

$$= p(1-p).$$

(6) $X^* = \max(X_1, \ldots, X_n)$ and $X_* = \min(X_1, \ldots, X_n)$

$X_1, \ldots, X_n$ iid $\{ p(x),\ x \in S \}$, $S \subseteq \mathbb{R}$

(6.a) X_*, X^* has the same support as X_i, $i = 1, \ldots, n$ $\Rightarrow S$

$$\begin{aligned}
&\mathbb{P}[X^* = x], \quad x \in S \\
&= \mathbb{P}[\max(X_1, \ldots, X_n) = x] \\
&= \mathbb{P}[X_1 \le x, X_2 \le x, \ldots, X_n \le x] - \mathbb{P}[X_1 < x, X_2 < x, \ldots, X_n < x] \\
&= \prod_{i=1}^{n} \mathbb{P}[X_i \le x] - \prod_{i=1}^{n} \mathbb{P}[X_i < x] \quad (\because X_i\text{'s are independent}) \\
&= (\mathbb{P}[X_1 \le x])^n - (\mathbb{P}(X_1 < x))^n \quad (\text{Indentical distributions}) \\
&= \Big(\sum_{y \in S : y \le x} p(y)\Big)^n - \Big(\sum_{y \in S : y < x} p(y)\Big)^n \quad //
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}[X_* = x], \quad x \in S \\
&= \mathbb{P}(\min(X_1, \ldots, X_n) = x] \\
&= \mathbb{P}[X_1 \ge x, X_2 \ge x, \ldots, X_n \ge x] - \mathbb{P}[X_1 > x, X_2 > x, \ldots, X_n > x] \\
&= \prod_{i=1}^{n} \mathbb{P}(X_i \ge x) - \prod_{i=1}^{n} \mathbb{P}[X_i > x] \\
&= [\mathbb{P}(X_1 \ge x)]^n - (\mathbb{P}[X_1 > x])^n \\
&= \Big(\sum_{y \in S : y \ge x} p(y)\Big)^n - \Big(\sum_{y \in S : y > x} p(y)\Big)^n \quad //
\end{aligned}$$

(6.b) $p(x) = a(1-a)^x$, $x = 0, 1, \ldots$

$$\begin{aligned}
&\mathbb{P}[X^* = x], \quad x = 0, 1, \ldots \\
&= \left[\sum_{y=0}^{x} a(1-a)^y\right]^n - \left[\sum_{y=0}^{x-1} a(1-a)^y\right]^n \\
&= \left[\frac{\cancel{a}(1 - (1-a)^{x+1})}{\cancel{1-(1-a)}}\right]^n - \left[\frac{\cancel{a}(1 - (1-a)^x)}{\cancel{1-(1-a)}}\right]^n \quad \left(\begin{array}{l}\text{By sum of} \\ \text{geometric series}\end{array}\right) \\
&= [1 - (1-a)^{x+1}]^n - [1 - (1-a)^x]^n \quad //
\end{aligned}$$

$$\mathbb{P}[X_\infty = x]$$

$$= \left(\sum_{y\in S:\, y\geq x} p(y)\right)^n - \left(\sum_{y\in S:\, y > x} p(y)\right)^n$$

$$= \left(\sum_{y=x}^{\infty} a(1-a)^y\right)^n - \left[\sum_{y=x+1}^{\infty} a(1-a)^y\right]^n$$

$$= \left[\frac{a(1-a)^x}{\cancel{1-(1-a)}}\right]^n - \left(\frac{a(1-a)^{x+1}}{\cancel{1-(1-a)}}\right)^n$$

$$= \left((1-a)^n\right)^x - (1-a)^n \cdot \left((1-a)^n\right)^x$$

$$= \left[1-(1-a)^n\right]\left((1-a)^n\right)^x$$

$\Rightarrow$ X_∞ is a geometric pmf on $\mathbb{N}$ with parameter $(1-(1-a)^n)$ //

(7) $X : \Omega \to \mathbb{R}$, discrete with $\mathbb{P}(X \in \mathbb{N}) = 1$

(7.a) $X \sim \text{Bin}(n; p)$, $n > 0$, $n \in \mathbb{N}_0$, $0 < p < 1$

$$\mathbb{E}\left[\frac{1}{1+X}\right] = \sum_{k=0}^{n} \frac{1}{1+k}\binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\,(k+1)!} p^k (1-p)^{n-k}$$

$$= \sum_{\ell=1}^{n+1} \frac{n!}{(n+1-\ell)!\,\ell!} p^{\ell-1} (1-p)^{n+1-\ell} \quad \left(\text{By subs. } k+1=\ell\right)$$

$$= \frac{1}{p(n+1)} \sum_{\ell=1}^{n+1} \frac{(n+1)!}{(n+1-\ell)!\,\ell!} p^{\ell} (1-p)^{(n+1)-\ell}$$

$$= \frac{1}{p(n+1)} \left[\sum_{\ell=0}^{n+1} \binom{n+1}{\ell} p^{\ell} (1-p)^{(n+1)-\ell} - (1-p)^{n+1}\right]$$

$$= \frac{1}{p(n+1)} \left[\left[p + (1-p)\right]^{n+1} - (1-p)^{n+1}\right]$$

$$= \frac{1}{p(n+1)} \left[1 - (1-p)^{n+1}\right] \quad //$$

(7.b) $X \sim \text{Pois}(\lambda)$, $\lambda > 0$

$$\mathbb{E}\left[\frac{1}{1+X}\right] = \sum_{k=0}^{\infty} \frac{1}{(1+k)} \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^{k+1}}{(k+1)!}$$

$$= \frac{1}{\lambda}\left[\sum_{\ell=1}^{\infty} \frac{e^{-\lambda}\lambda^{\ell}}{\ell!}\right] \quad (\text{by } \ell = k+1)$$

$$= \frac{1}{\lambda}\left[\sum_{\ell=0}^{\infty} \frac{e^{-\lambda}\lambda^{k}}{\ell!} - e^{-\lambda}\right]$$

$$= \frac{1}{\lambda}\left[1 - e^{-\lambda}\right] \quad //$$

(8) $\{X_1, \ldots, X_n ; \nu\} \; \Omega \to \mathbb{R}$ are Mutually Independent.

$$\mathbb{P}[X_i = x] = p(x), \quad x \in S, \quad i = 1, \ldots, n$$

$$\mathbb{P}[\nu \in \{0, 1, \ldots, N\}] = 1$$

Method I:

$$\sum_{k=1}^{\nu} X_k = \sum_{k=1}^{\infty} X_k \, \mathbb{1}[\nu > k-1]$$

$$\therefore \mathbb{E}\left[\sum_{k=1}^{\nu} X_k\right] = \mathbb{E}\left[\sum_{k=1}^{\infty} X_k \, \mathbb{1}[\nu > k-1]\right]$$

$$= \sum_{k=1}^{\infty} \mathbb{E}\left[X_k \, \mathbb{1}[\nu > k-1]\right]$$

$$= \sum_{k=1}^{\infty} \mathbb{E}(X_k) \cdot \mathbb{P}[\nu > k-1] \quad (\nu \perp\!\!\!\perp X_k, \; k = 1, 2.$$

$$= \mathbb{E}[X_1] \sum_{k=1}^{\infty} \mathbb{P}[\nu > k-1] \quad \left(\mathbb{E}(X_k) = \mathbb{E}(X_1) \text{ for } \forall k, \text{ iid}\right)$$

$$= \mathbb{E}[X_1] \sum_{n=0}^{\infty} \mathbb{P}[\nu > n] \quad (\text{by } n = k-1)$$

$$= \mathbb{E}[X_1] \, \mathbb{E}[\nu] \quad (\text{Using Problem 4})$$

Method II
(Using Iterated Conditioning)

$$\mathbb{E}\left[\sum_{k=1}^{\nu} X_k\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{k=1}^{\nu} X_k \,\middle|\, \nu\right]\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{\nu} \mathbb{E}(X_k)\right] \quad (\text{Since } \nu \perp\!\!\!\perp X_1, \ldots, X_n)$$

$$= \mathbb{E}\left[\nu \cdot \mathbb{E}(X_1)\right] \quad (\text{Since } X_k\text{'s are identically distributed})$$

$$= \mathbb{E}(X_1) \cdot \mathbb{E}(\nu)$$

(9) $\mathbb{P}[X = x] = \mathbb{P}(Y = y) = \frac{1}{10}$, $x, y = 0, \ldots, 9$

$X + Y = \xi \cdot 10 + \eta$,
$\mathbb{P}[\xi \in \{0, \ldots, 4\}] = \mathbb{P}(\eta \in \{0, \ldots, 9\}] = 1$

(9.a) Given that $S_X = S_Y = \{0, 1, \ldots, 9\}$, the support of the random variable $X + Y$ is, $S_{X+Y} = \{0, 1, \ldots, 18\}$.

First, observe the pmf of $X+Y$
for $k \in S_{X+Y}$,

$$\mathbb{P}[X + Y = k] = \mathbb{P}[Y = k - X]$$
$$= \sum_{x \in S_X} \mathbb{P}[Y = k - X, X = x]$$
$$= \sum_{x=0}^{9} \mathbb{P}(Y = k - x, X = x]$$

For $0 \le k \le 9$,

$$\mathbb{P}(X + Y = k] = \sum_{x=0}^{k} \mathbb{P}[Y = k - x, X = x] \quad \left(\because \mathbb{P}(Y = k - x) = 0, \ x > k \right)$$
$$= \sum_{x=0}^{k} \mathbb{P}(Y = k - x) \cdot \mathbb{P}(X = x) \quad (X \perp\!\!\!\perp Y)$$
$$= \sum_{x=0}^{k} \left(\frac{1}{10}\right)^2$$
$$= \frac{k+1}{100}$$

For $9 < k \le 18$,

$$\mathbb{P}(X + Y = k] = \sum_{x=k-9}^{9} \mathbb{P}[Y = k - x, X = x] \quad \left(\because \text{for } x < k - 9, \ \mathbb{P}(Y = k - x) = 0 \right)$$
$$= \sum_{x=k-9}^{9} \mathbb{P}(Y = k - x) \cdot \mathbb{P}(X = x) \quad (X \perp\!\!\!\perp Y)$$
$$= \sum_{x=k-9}^{9} \left(\frac{1}{10}\right)^2$$
$$= \frac{19-k}{100}$$

$$\therefore \mathbb{P}(X + Y = k] = \begin{cases} (k+1)/100, & 0 \le k \le 9 \\ (19-k)/100, & 9 < k \le 18 \\ 0, & \text{elsewhere} \end{cases}$$

$$X + Y = 10 \cdot \xi + \eta$$

$S_{X+Y} = \{0, 1, \ldots, 18\}$, $S_\xi = S_\eta = \{0, 1, \ldots, 9\}$

Hence, it is clear that $\eta \in \{0, 1, \ldots, 9\}$ and $\xi \in \{0, 1\}$ (the unit and the ten's places of the values of $X+Y$)

* Note that there is a one to one correspondence between the pair (ξ, η) and the rv $(X+Y)$.

Hence, for $x \in \{0, 1\}$ and $y \in \{0, 1, \ldots, 9\}$

$$\mathbb{P}[\xi = x, \eta = y] = \mathbb{P}[X + Y = 10x + y]$$

$$= \begin{cases} (y+1)/100, & x = 0 \\ (19-(10+y))/100, & x = 1, y \neq 9 \end{cases}$$

$$= \begin{cases} (y+1)/100, & x = 0, y \in \{0, 1, \ldots, 9\} \\ (9-y)/100, & x = 1, y \in \{0, 1, \ldots, 8\} \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

(4b) $\mathbb{P}[\xi = 0, \eta = 0] = 1/100$ // (substituting $x = y = 0$ in (1))

$$\mathbb{P}[\eta = 0] = \mathbb{P}[\xi = 0, \eta = 0] + \mathbb{P}[\xi = 1, \eta = 0]$$
$$= 1/100 + 9/100 = 1/10 \text{ //}$$

$$\mathbb{P}[\xi = 0] = \sum_{y=0}^{9} (y+1)/100$$
$$= \frac{1}{100}\left[\frac{9 \times 10}{2} + 10\right] \quad \left[\text{Using } \sum_{y=0}^{n} y = \frac{n(n+1)}{2}\right]$$
$$= 0.55 \text{ //}$$

* Note that $\mathbb{P}[\xi = 0, \eta = 0] \neq \mathbb{P}[\xi = 0]\,\mathbb{P}[\eta = 0]$

$\Rightarrow \xi \not\perp\!\!\!\perp \eta$ (Not independent!)

(10) $\mathbb{P}[v=k] = 1/N, \quad k=1,2,\dots,N$

$$\mathbb{P}(X=x \mid v=k) = \begin{cases} 1/(k+1), & 0 \le x \le k \\ 0, & \text{otherwise.} \end{cases} \qquad \text{—(1)}$$

(10a) $\mathbb{P}[v=k \mid X<j] \qquad k=1,2,\dots,N, \quad j=1,2,\dots,N$

$$= \frac{\mathbb{P}(v=k, X<j)}{\mathbb{P}[X<j]}$$

$$= \frac{\mathbb{P}[X<j \mid v=k]\,\mathbb{P}[v=k]}{\sum_{l=1}^{N}\mathbb{P}[X<j \mid v=l]\,\mathbb{P}[v=l]}$$

$$= \frac{\mathbb{P}[X<j \mid v=k]\,\mathbb{P}[v=k]}{\sum_{l=1}^{j-1}\mathbb{P}[X<j \mid v=l]\,\mathbb{P}[v=l] + \sum_{l=j}^{N}\mathbb{P}[X<j \mid v=l]\,\mathbb{P}[v=l]}$$

$$= \frac{\mathbb{P}(X<j \mid v=k)\cdot \cancel{1/N}}{\sum_{l=1}^{j-1}(1)\left(\cancel{1/N}\right) + \sum_{l=j}^{N}\left(\frac{j}{l+1}\right)\cdot\cancel{\left(\frac{1}{N}\right)}} \qquad (\text{from } (1))$$

$$= \frac{\mathbb{P}(X<j \mid v=k)}{(j-1) + j\sum_{l=j}^{N}\frac{1}{(l+1)}}$$

$$= \begin{cases} \dfrac{1}{j-1+j\sum_{l=j}^{N}\left(\frac{1}{l+1}\right)}, & \text{if } j>k \\[2ex] \dfrac{j/(k+1)}{j-1+j\sum_{l=j}^{N}\left(\frac{1}{l+1}\right)} & \text{if } k \geq j \end{cases}$$

//

(10b) $\mathbb{E}[v=k \mid X<j], \quad j=1,\dots,N$

$=$

(10.c) $\mathbb{E}(X) = \sum_{x=0}^{N} x \, \mathbb{P}[X = x]$

$$= \sum_{x=0}^{N} x \sum_{a=1}^{N} \mathbb{P}(X = x \mid v = a) \underbrace{\mathbb{P}(v = a)}_{1/N}$$

$$= \frac{1}{N} \sum_{a=1}^{N} \sum_{x=0}^{N} x . \mathbb{P}(X = x \mid v = a)$$

$$= \frac{1}{N} \sum_{a=1}^{N} \sum_{x=0}^{a} x . \mathbb{P}[X = x \mid v = a] \qquad \left[\mathbb{P}[X = x \mid v = a] = 0, \text{ when } x > a \right]$$

$$= \frac{1}{N} \sum_{a=1}^{N} \sum_{x=0}^{a} x \times \frac{1}{(a+1)}$$

$$= \frac{1}{N} \sum_{a=1}^{N} \frac{1}{(a+1)} \left(\sum_{x=0}^{a} x \right)$$

$$= \frac{1}{N} \sum_{a=1}^{N} \frac{1}{(a+1)} \cdot \frac{a(a+1)}{2} \qquad (\text{Sum of the arithmetic series})$$

$$= \frac{1}{2N} \sum_{a=1}^{N} a$$

$$= \frac{1}{2N} \cdot \frac{N(N+1)}{2}$$

$$= \frac{(N+1)}{4} \quad //$$