
ENEE 324 – 01*
SPRING 2018
ENGINEERING PROBABILITY

ANSWER KEY TO TEST # 1:

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a. A natural way to encode the outcome ω of this experiment is through the pair (Choice, Color) with

$$\omega = (\text{Choice of urn, Color}) \in \Omega \equiv \{1,2,3\} \times \{\text{Red, Blue}\}.$$

Obviously, $\mathcal{F} = \mathcal{P}(\Omega)$. Moreover, the random urn selection translates into

$$\mathbb{P}\left[\text{Select urn } c\right] = \mathbb{P}\left[\{c\} \times \{\text{Red, Blue}\}\right] = \frac{1}{3}, \quad c = 1,2,3.$$

Also, the statement of the problem yields

$$\mathbb{P}\left[\text{A red ball is drawn}|\text{Select urn } c\right] = \frac{R_c}{R_c + B_c}, \quad c = 1,2,3.$$

Note that $B_3 = 0$.
It follows that

$$\begin{aligned}
\mathbb{P}\left[\{c, \text{Red}\}\right] &= \mathbb{P}\left[[\text{Select urn } c] \cap [\text{A red ball is drawn}]\right] \\
&= \mathbb{P}\left[\text{A red ball is drawn}|\text{Select urn } c\right] \cdot \mathbb{P}\left[\text{Select urn } c\right] \\
&= \frac{1}{3} \cdot \frac{R_c}{R_c + B_c}, \quad c = 1,2,3. \qquad (1.1)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{P}\left[\{c, \text{Blue}\}\right] &= \mathbb{P}\left[\text{Select urn } c\right] - \mathbb{P}\left[\{c, \text{Red}\}\right] \\
&= \frac{1}{3} - \frac{1}{3} \cdot \frac{R_c}{R_c + B_c} \\
&= \frac{1}{3} \cdot \frac{B_c}{R_c + B_c}, \quad c = 1,2,3. \qquad (1.2)
\end{aligned}$$

Thus, $\mathbb{P}\left[\{\omega\}\right]$ is specified for each ω in Ω!

b. We readily see that

$$\begin{aligned} & \mathbb{P}\left[\text{Ball drawn selected from urn } U_1 | \text{Ball drawn is red}\right] \\ = & \frac{\mathbb{P}\left[\text{Ball drawn selected from urn } U_1, \text{ball drawn is red}\right]}{\mathbb{P}\left[\text{Ball drawn is red}\right]} \\ = & \frac{\mathbb{P}\left[\{1, \text{Red}\}\right]}{\mathbb{P}\left[\{1, \text{Red}\}\right] + \mathbb{P}\left[\{2, \text{Red}\} + \mathbb{P}\left[\{3, \text{Red}\}\right]\right]} \\ = & \frac{\frac{1}{3} \cdot \frac{R_1}{R_1+B_1}}{\frac{1}{3} \cdot \frac{R_1}{R_1+B_1} + \frac{1}{3} \cdot \frac{R_2}{R_2+B_2} + \frac{1}{3} \cdot \frac{R_3}{R_3+B_3}} \\ = & \frac{\frac{R_1}{R_1+B_1}}{\frac{R_1}{R_1+B_1} + \frac{R_2}{R_2+B_2} + 1}. \end{aligned} \quad (1.3)$$

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a. Note that

$$\mathbb{P}\left[X = x\right] = \frac{1}{2a+1}, \quad x = -a, \ldots, -1, 0, 1, \ldots a$$

so that

$$\mathbb{E}\left[X\right] = \sum_{x=-a}^{a} x \cdot \mathbb{P}\left[X = x\right] = \frac{1}{2a+1} \sum_{x=-a}^{a} x = 0$$

by symmetry

b. Obviously, the rv Y is a discrete rv since $Y = g(X)$ with $g : \mathbb{R} \to \mathbb{R} : x \to |x|$ and the rv X is discrete! Its support is $\{0, 1, \ldots, a\}$. First, since $Y = 0$ if and only if $X = 0$, we get

$$\mathbb{P}\left[Y = 0\right] = \mathbb{P}\left[X = 0\right] = \frac{1}{2a+1}$$

and for $y = 1, \ldots, a$, since $Y = y$ if and only if either $X = y$ or $X = -y$, we have

$$\mathbb{P}\left[Y = y\right] = \mathbb{P}\left[X = -y\right] + \mathbb{P}\left[X = y\right] = \frac{2}{2a+1}.$$

c. It follows that

$$\begin{aligned} \mathbb{E}\left[Y\right] &= \sum_{y=0}^{a} y \cdot \mathbb{P}\left[Y = y\right] \\ &= \sum_{y=1}^{a} y \cdot \frac{2}{2a+1} \\ &= \frac{2}{2a+1} \cdot \sum_{y=1}^{a} y \\ &= \frac{2}{2a+1} \cdot \frac{a(a+1)}{2} = \frac{a(a+1)}{2a+1}. \end{aligned} \quad (1.4)$$

c. Obviously $|\mathbb{E}\left[X\right]| = 0$ while $\mathbb{E}\left[|X|\right] > 0$!

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a. Obviously

$$\begin{aligned}
\mathbb{P}[X=Y] &= \sum_{x=0}^{\infty} \mathbb{P}[X=Y, X=x] \\
&= \sum_{x=0}^{\infty} \mathbb{P}[X=x, Y=x] \\
&= \sum_{x=0}^{\infty} \mathbb{P}[X=x]\,\mathbb{P}[Y=x] \quad \text{[By the independence of the rvs } X \text{ and } Y\text{]} \\
&= \sum_{x=0}^{\infty} (1-a)^x a \cdot (1-b)^x b \\
&= \frac{ab}{1-(1-a)(1-b)} \qquad (1.5)
\end{aligned}$$

and we conclude that

$$\begin{aligned}
\mathbb{P}[X \neq Y] &= 1 - \mathbb{P}[X=Y] \\
&= 1 - \frac{ab}{1-(1-a)(1-b)} \\
&= \frac{a+b-2ab}{1-(1-a)(1-b)} = \frac{a(1-b)+b(1-a)}{1-(1-a)(1-b)}. \qquad (1.6)
\end{aligned}$$

b. We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{x=0} x \cdot a(1-a)^x \\
&= (1-a)\left(\sum_{x=1} x \cdot a(1-a)^{x-1}\right) \\
&= \frac{1-a}{a} \qquad (1.7)
\end{aligned}$$

as we identify $\sum_{x=1} x \cdot a(1-a)^{x-1}$ to be the expectation of the geometric pmf on $\mathbb{N}_0$, whose value was computed in class to be a^{-1}. In a similar way we have $\mathbb{E}[Y] = (1-b)b^{-1}$.

c. Finally, with mapping $g : \mathbb{R}^2 \to \mathbb{R} : (x,y) \to xy$, we have

$$\begin{aligned}
\mathbb{E}[XY] &= \mathbb{E}[g(X,Y)] \\
&= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} g(x,y)\mathbb{P}[X=x, Y=y] \\
&= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} xy \cdot \mathbb{P}[X=x]\,\mathbb{P}[Y=y] \quad \text{[By the independence of the rvs } X \text{ and } Y\text{]} \\
&= \sum_{x=0}^{\infty}\left(\sum_{y=0}^{\infty} y \cdot \mathbb{P}[Y=y]\right) x \cdot \mathbb{P}[X=x] \\
&= \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{(1-a)(1-b)}{ab}. \qquad (1.8)
\end{aligned}$$

4.

a. Label the items in the shipment, say $1, \ldots, 50$, and take Ω to be the collection of all subsets of size 5 drawn from $\{1, \ldots, 50\}$ – Clearly the order does not matter and $|\Omega| = \binom{50}{5}$. We have $\mathcal{F} = \mathcal{P}(\Omega)$ and the probability assignment is the uniform probability assignement, so

$$\mathbb{P}\left[\{\omega\}\right] = \frac{1}{\binom{50}{5}}, \quad \omega \in \Omega.$$

b. If the shipment contains five (5) defectives, then the shipment will be accepted if either the sample contains no defective item – There are $\binom{45}{5}$ such possibilities, or if the sample contains exactly one defective item – There are $5 \cdot \binom{45}{4}$ such possibilities. Therefore,

$$p_5 = \frac{\binom{45}{5} + 5 \cdot \binom{45}{4}}{\binom{50}{5}}.$$

c. If the shipment contains fifteen (15) defectives, then the shipment will be accepted if either the sample contains no defective item – There are $\binom{35}{5}$ such possibilities, or if the sample contains exactly one defective item – There are $15 \cdot \binom{35}{4}$ such possibilities. Therefore,

$$p_{15} = \frac{\binom{35}{5} + 15 \cdot \binom{35}{4}}{\binom{50}{5}}.$$