ENEE 324 – 01*
SPRING 2018
ENGINEERING PROBABILITY

ANSWER KEY TO TEST # 2:

1

a. Note that

$$\begin{aligned} f_X(x) &= \int_{\mathbb{R}} f_{X,Y}(x,y)dy \\ &= \begin{cases} 0 & \text{if } x < 0 \\ \int_x^\infty 2e^{-(x+y)}dy & \text{if } x \geq 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } x < 0 \\ 2e^{-2x} & \text{if } x \geq 0 \end{cases} \end{aligned} \tag{1.1}$$

so $X \sim \text{Exp}(2)$, while

$$\begin{aligned} f_Y(y) &= \int_{\mathbb{R}} f_{X,Y}(x,y)dx \\ &= \begin{cases} 0 & \text{if } y < 0 \\ \int_0^y 2e^{-(x+y)}dx & \text{if } y \geq 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } y < 0 \\ 2e^{-y}\left(1 - e^{-y}\right) & \text{if } y \geq 0. \end{cases} \end{aligned} \tag{1.2}$$

b. Note that

$$\mathbb{E}\left[X - Y\right] = \mathbb{E}\left[X\right] - \mathbb{E}\left[Y\right] = \frac{1}{2} - \mathbb{E}\left[Y\right]$$

with

$$\begin{aligned} \mathbb{E}\left[Y\right] &= \int_0^\infty y f_Y(y)dy \\ &= 2\int_0^\infty ye^{-y}\left(1 - e^{-y}\right)dy \\ &= 2\int_0^\infty ye^{-y}dy - \int_0^\infty 2ye^{-2y}dy \\ &= 2 \cdot 1 - \frac{1}{2}. \end{aligned} \tag{1.3}$$

Therefore,

$$\mathbb{E}\left[X-Y\right]=\frac{1}{2}-2+\frac{1}{2}=-1.$$

c. We need only consider the case $x \geq 0$, and on that range we have

$$f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}=\begin{cases}0 & \text{if } y<x\\ \frac{2e^{-(x+y)}}{2e^{-2x}}=e^{-(y-x)} & \text{if } x\leq y\end{cases}$$

whence

$$\mathbb{E}\left[Y|X=x\right]=\int_x^\infty ye^{-(y-x)}dy=e^x\cdot\int_x^\infty ye^{-y}dy,\quad x\geq 0.$$

Integration by parts gives

$$\begin{aligned}\int_x^\infty ye^{-y}dy &= \int_x^\infty y\left(-e^{-y}\right)'dy\\ &= \left(-ye^{-y}\right)_x^\infty+\int_x^\infty e^{-y}dy\\ &= xe^{-x}+e^{-x}\end{aligned}\tag{1.4}$$

whence

$$\mathbb{E}\left[Y|X=x\right]=1+x,\quad x\geq 0.$$

2. ____

Most of the arguments made here rely on the fact that

$$\int_{\mathbb{R}} e^{-\frac{1}{2}\left(\frac{t-a}{\sigma}\right)^2}dt=\sqrt{2\pi\sigma^2},\qquad \begin{matrix}a\in\mathbb{R}\\ \sigma>0.\end{matrix}$$

a. We have

$$\begin{aligned}1 &= \int_{\mathbb{R}^2} f_{X,Y}(x,y)dxdy\\ &= A\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2}}dy\right)e^{-\frac{x^2}{2}}dx\\ &= A\int_{\mathbb{R}}\sqrt{2\pi}e^{-\frac{x^2}{2}}dx\\ &= A\left(\sqrt{2\pi}\right)^2\quad \text{whence } A=\tfrac{1}{2\pi}.\end{aligned}\tag{1.5}$$

b. First,

$$\begin{aligned}f_X(x) &= \int_{\mathbb{R}} f_{X,Y}(x,y)dy\\ &= Ae^{-\frac{x^2}{2}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2}}dy\\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\quad x\in\mathbb{R}.\end{aligned}\tag{1.6}$$

Similarly,

$$\begin{aligned} f_Y(y) &= \int_{\mathbb{R}} f_{X,Y}(x,y)dx \\ &= A\int_{\mathbb{R}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(x-y)^2}{2}} dx, \quad y \in \mathbb{R} \end{aligned} \tag{1.7}$$

with

$$\begin{aligned} \frac{x^2}{2} + \frac{(x-y)^2}{2} &= \frac{1}{2}\left(x^2 + x^2 - 2xy + y^2\right) \\ &= \frac{1}{2}\left((\sqrt{2}x)^2 - 2(\sqrt{2}x)\frac{y}{\sqrt{2}} + y^2\right) \\ &= \frac{1}{2}\left(\left[\sqrt{2}x - \frac{y}{\sqrt{2}}\right]^2 + \left[\frac{y}{\sqrt{2}}\right]^2\right) \\ &= \frac{1}{2}\left(2\left[x - \frac{y}{2}\right]^2 + \left[\frac{y}{\sqrt{2}}\right]^2\right) \end{aligned} \tag{1.8}$$

by a completion of squares argument. Thus,

$$\begin{aligned} A\int_{\mathbb{R}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(x-y)^2}{2}} dx &= A\int_{\mathbb{R}} e^{-\frac{1}{2}\left(2\left[x-\frac{y}{2}\right]^2 + \left[\frac{y}{\sqrt{2}}\right]^2\right)} dx \\ &= Ae^{-\frac{y^2}{4}} \cdot \int_{\mathbb{R}} e^{-\frac{1}{2}\frac{(x-\frac{y}{2})^2}{\frac{1}{2}}} dx \\ &= Ae^{-\frac{y^2}{4}} \cdot \sqrt{2\pi \cdot \frac{1}{2}} \end{aligned} \tag{1.9}$$

whence

$$f_Y(y) = \frac{1}{\sqrt{4\pi}} e^{-y^2}, \quad y \in \mathbb{R}.$$

It follows that $X \sim \mathrm{N}(0,1)$ and $Y \sim \mathrm{N}(0,2)$.

c. Next, for each y in $\mathbb{R}$, it holds that

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{\frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(x-y)^2}{2}}}{\frac{1}{\sqrt{4\pi}} e^{-\frac{1}{2}\left(\frac{y}{\sqrt{2}}\right)^2}} \\ &= \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}(\dots)}, \quad x \in \mathbb{R} \end{aligned} \tag{1.10}$$

with

$$\begin{aligned} \ldots &= x^2 + (x-y)^2 - \frac{y^2}{2} \\ &= 2x^2 - 2xy + \frac{y^2}{2} \\ &= \left(\sqrt{2}x - \frac{y}{\sqrt{2}}\right)^2 \\ &= 2\left(x - \frac{y}{2}\right)^2, \quad x \in \mathbb{R}. \end{aligned} \tag{1.11}$$

Thus,

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{\pi}} e^{-\left(x-\frac{y}{2}\right)^2}, \quad x \in \mathbb{R}$$

and it is now plain that the conditional distribution of X given that $Y = y$ is a normal distribution $\mathrm{N}(\frac{y}{2}, \frac{1}{2})$.

d. This time we have

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{\frac{1}{2\pi} e^{-\frac{x^2}{2}} \cdot e^{-\frac{(x-y)^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}}, \end{aligned} \tag{1.12}$$

and the conditional distribution of Y given that $X = x$ is a normal distribution $\mathrm{N}(x, 1)$, hence $\mathbb{E}[Y|X = x] = x$.

3.

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a. Note that the conditional probability distribution of the rv X given $B = \text{Head}$ (resp. $B = \text{Tail}$) coincides with that of U (resp. V). Thus, it holds that

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X|B = \text{Head}]\,\mathbb{P}[B = \text{Head}] + \mathbb{E}[X|B = \text{Tail}]\,\mathbb{P}[B = \text{Tail}] \\ &= \mathbb{E}[U]\,\mathbb{P}[B = \text{Head}] + \mathbb{E}[V]\,\mathbb{P}[B = \text{Tail}] \\ &= \frac{1}{2}p + 0 \cdot (1-p) = \frac{p}{2}. \end{aligned} \tag{1.13}$$

b. With x in $\mathbb{R}$, we have

$$\begin{aligned} &\mathbb{P}[B = \text{Head}|X = x] \\ &\quad = \frac{f_{X|B}(x|\text{Head})\mathbb{P}[B = \text{Head}]}{f_X(x)} \\ &\quad = \frac{pf_U(x)}{pf_U(x) + (1-p)f_V(x)} \\ &\quad = \begin{cases} 0 & \text{if } x < 0 \\ \frac{p}{p + (1-p)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } 1 < x \end{cases} \end{aligned} \tag{1.14}$$

with an obvious notation.

c. We start with the observation that the rv X can be represented as

$$\begin{aligned} X &= \mathbf{1}[B = \text{Head}] \cdot U + \mathbf{1}[B = \text{Tail}] \cdot V \\ &= B^\star \cdot U + (1 - B^\star) \cdot V \end{aligned} \tag{1.15}$$

with $B^\star$ being the indicator rv $B^\star = \mathbf{1}[B = \text{Head}]$ and $\mathbf{1}[B = \text{Tail}] = 1 - B^\star$. Therefore, we get

$$\begin{aligned} \text{Cov}[X, U] &= \text{Cov}[B^\star \cdot U + (1 - B^\star) \cdot V, U] \\ &= \text{Cov}[B^\star \cdot U, U] + \text{Cov}[(1 - B^\star) \cdot V, U] \end{aligned} \tag{1.16}$$

The rvs $B^\star$, U and V being mutually independent,

$$\begin{aligned} \text{Cov}[B^\star \cdot U, U] &= \mathbb{E}\left[B^\star \cdot U^2\right] - \mathbb{E}[B^\star \cdot U]\,\mathbb{E}[U] \\ &= \mathbb{E}[B^\star]\,\mathbb{E}\left[U^2\right] - \mathbb{E}[B^\star]\,\mathbb{E}[U]\,\mathbb{E}[U] \\ &= p\text{Var}[U] \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} \text{Cov}[(1 - B^\star) \cdot V, U] &= \mathbb{E}[(1 - B^\star) \cdot UV] - \mathbb{E}[(1 - B^\star) \cdot V]\,\mathbb{E}[U] \\ &= \mathbb{E}[1 - B^\star]\,\mathbb{E}[UV] - \mathbb{E}[1 - B^\star]\,\mathbb{E}[V]\,\mathbb{E}[U] \\ &= (1 - p)\text{Cov}[U, V] = 0 \end{aligned} \tag{1.18}$$

since the rvs U and V are independent (hence uncorrelated). Thus,

$$\text{Cov}[X, U] = p\text{Var}[U] = p\left(\frac{1}{3} - \left(\frac{1}{2}\right)^2\right) = \frac{p}{12} \neq 0!$$

and the rvs X and U are clearly correlated.

4.

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a. The rvs X and Y are clearly independent since

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y), \quad x, y \in \mathbb{R}$$

as we note that

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y)dy = \begin{cases} xe^{-\frac{x^2}{2}} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and similarly,

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)dx = \begin{cases} ye^{-\frac{y^2}{2}} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0. \end{cases}$$

b. Obviously, since $(xe^{-\frac{x^2}{2}}) = (-e^{-\frac{x^2}{2}})'$, we get

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi = \begin{cases} 0 & \text{if } x < 0 \\ \\ 1 - e^{-\frac{x^2}{2}} & \text{if } x \geq 0 \end{cases} \tag{1.19}$$

and

$$F_Y(y) = \int_{-\infty}^{y} f_X(\eta) d\eta = \begin{cases} 0 & \text{if } y < 0 \\ \\ 1 - e^{-\frac{y^2}{2}} & \text{if } y \geq 0. \end{cases} \tag{1.20}$$

c. Finally, with $r > 0$, we have

$$\begin{aligned} \mathbb{P}\left[R \leq r\right] &= \mathbb{P}\left[Y > 0, \frac{X}{Y} \leq r\right] + \mathbb{P}\left[Y \leq 0, \frac{X}{Y} \leq r\right] \\ &= \mathbb{P}\left[Y > 0, X \leq rY\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left[Y > 0\right] \mathbf{1}\left[X \leq rY\right] | Y\right]\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}\left[\mathbf{1}\left[Y > 0\right] \mathbf{1}\left[X \leq rY\right] | Y = y\right]\right)_{y=Y}\right] \\ &= \mathbb{E}\left[\left(\mathbb{E}\left[\mathbf{1}\left[y > 0\right] \mathbf{1}\left[X \leq ry\right] | Y = y\right]\right)_{y=Y}\right] \\ &= \mathbb{E}\left[\mathbf{1}\left[Y > 0\right] \left(\mathbb{E}\left[\mathbf{1}\left[X \leq ry\right] | Y = y\right]\right)_{y=Y}\right] \\ &= \mathbb{E}\left[\mathbf{1}\left[Y > 0\right] \left(\mathbb{E}\left[\mathbf{1}\left[X \leq ry\right]\right]\right)_{y=Y}\right] \\ &= \mathbb{E}\left[\mathbf{1}\left[Y > 0\right] \mathbb{P}\left[X \leq ry\right]_{y=Y}\right] \\ &= \mathbb{E}\left[\mathbf{1}\left[Y > 0\right] \left(1 - e^{-\frac{r^2 Y^2}{2}}\right)\right] \\ &= \int_0^\infty \left(1 - e^{-\frac{r^2 \eta^2}{2}}\right) \eta e^{-\frac{\eta^2}{2}} d\eta \\ &= 1 - \int_0^\infty \eta e^{-\frac{(1+r^2)\eta^2}{2}} d\eta \\ &= 1 - \frac{1}{1+r^2}, \quad r \geq 0. \end{aligned} \tag{1.21}$$

A second solution:

$$\begin{aligned} \mathbb{P}\left[R \leq r\right] &= \mathbb{P}\left[Y > 0, X \leq rY\right] \\ &= \int_{\{(x,y) \in \mathbb{R}_+^2 :\ x \leq ry\}} f_X(x) f_Y(y) dx dy \\ &= \int_0^\infty \left(\int_0^{ry} f_X(y) dx\right) f_Y(y) dy \\ &= \int_0^\infty \left(1 - e^{-\frac{r^2 y^2}{2}}\right) f_Y(y) dy \\ &= 1 - \frac{1}{1+r^2}, \quad r \geq 0. \end{aligned} \tag{1.22}$$