ENEE 324 Engineering Probability Lecture 2

Counting

For the case of rolling a single *fair* die, let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and let

$$\mathcal{A} = 2^{\Omega} = \{\emptyset, \Omega, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{2,3\}...\}$$

There are $2^6 = 64$ members in $\mathcal{A}$, and only 6 of these are elementary events. Declare that all singletons (elementary events) are *equally likely.* [recall fairness assumption]

Since $P(\Omega) = 1$ and Ω is the union of 6 singletons, all equally likely, it follows from the addition axiom that,

$$\text{Probability of a singleton} = \tfrac{1}{6}.$$

The probabilities of all the other events in $\mathcal{A}$ can be determined from this one fact ! We simply apply the addition axiom.

In a deck of *well-shuffled* cards, the probability of drawing the heart $= \frac{1}{52}$.

Example 1: Toss a coin repeatedly until the first head. In each toss,

$$P\{H\} = p \ : \ P\{T\} = 1 - p = q.$$

$\Omega = \{1, 2, 3, ...\}$ = sample space of # tosses needed until first head.
Assume $p \neq 0$.

$$\begin{aligned} p_j &= p\{j \text{ tosses until first head}\} \\ &= q^{j-1} \cdot p \qquad j = 1, 2, ... \end{aligned}$$

Where did this come from?

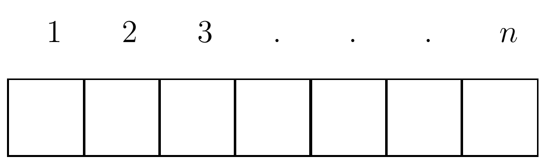
$$
\begin{aligned}
\sum_{j=1}^{\infty} q^{j-1}p &= p \cdot (1 + q + q^2 + ...) \\
&= p \cdot \lim_{n\to\infty} (1 + q + q^2 + ... + q^n) \\
&= p \cdot \lim_{n \to \infty} \frac{1 - q^n}{1 - q} \\
&= \frac{p}{p} \qquad \text{(because } q < 1\text{)} \\
&= 1
\end{aligned}
$$

Example 2: Lifetime of computer memory chip satisfies: "proportion of chips whose lifetime exceeds t decreases exponentially at the rate α."
Here $\alpha > 0$.

$$
\begin{aligned}
\Omega &= (0, \infty) \\
P[(t, \infty)] &= e^{-\alpha t} && t > 0 \\
P[(0, \infty)] &= e^{-0\alpha} && = 1 \text{ as it should be.} \\
P[(r, s)] &= P[(r, \infty)] - P[(s, \infty)] && = e^{-\alpha r} - e^{-\alpha s} \ , \ r < s
\end{aligned}
$$

Some combinatorics

(a) Given n distinct things, how many ways can we permute them?
Think of this as filling n marked cells



Fill cell 1 in any of n ways.
Fill cell 2 in any of $(n - 1)$ ways (with the remaining $(n - 1)$ things).
Fill cell 3 with any of $(n - 2)$ ways.

Fill cell n in (1) way.

Total number of ways of filling cells is $nPn = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$
We call this $n!$.

(b) Given n distinct things, how many different **permutations** of r things can we make from these n things?

Treat the problem as one of filling r out of n cells.

Proceeding as before we get

$$\begin{aligned} nPr &= n(n-1)(n-2)...(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

SAMPLING
(c) Given n distinct things, how many **combinations** of r things out of these n things can we make?
Denote this yet to be determined quantity as nCr.
Combinations ignore order. Thus,

$$\begin{aligned} nCr \ \cdot \ r! &= nPr \\ &= \frac{n!}{(n-r)!} \ . \\ \text{H}ence \quad nCr &= \frac{n!}{(n-r)!r!} \end{aligned}$$

The sampling is said to be **random** if all of these combinations are equally

likely. So the probability of a particular combination being picked up in a random sample is $\frac{(n-r)!r!}{n!}$
It is common to use the notation $\binom{n}{r}$ instead of nCr. These integers have a long history. **Newton's binomial expansion** says

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Proof: $(a+b)^n$ is an expression, homogeneous of degree n. Hence each term in $(a+b)^n$ will be of the form $a^k b^{n-k}$. How many are of this form? $\binom{n}{k}$ □

Identitites

$$\text{(i)} \quad \binom{n}{r} = \binom{n}{n-r}$$

$$\text{(ii)} \quad \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Single out an object – a_k, say.
Numbers of choices of r objects out of n objects = (number of choices that exclude a_k) + (number of choices that include a_k) = $\binom{n-1}{r} + \binom{n-1}{r-1}$

Example 3: (sample without replacement)

Total of N items.
Choose n at random without replacement.
This will yield $\binom{N}{n}$ possible samples.

If the N items are made up of r_1 blues and r_2 reds, $r_1 + r_2 = N$, then the probability of choosing *exactly* s_1 blues and $(n - s_1)$ reds, (here $s_1 \le n$ and

$s_1 \le r_1$, $(n - s_1) \le r_2$), is given by

$$\frac{\binom{r_1}{s_1}\binom{r_2}{n-s_1}}{\binom{N}{n}}$$

We call this the **hypergeometric law**
Where did this come from?
Answer: Think of each sample as equally likely and count how many there are favorable to the event of interest.

Example 4: (inspection for quality control) A batch of 100 manufactured items is checked by an inspector, who examines 10 items selected at random. If *none* of the 10 items is defective, the batch of 100 is accepted. Otherwise, the batch is subject to further inspection. What is the probability that a batch containing 10 defectives is accepted?

Solution: Number of ways of selecting 10 items of a batch of 100 is

$$N = \binom{100}{10}.$$

All such samples are equally likely.

A = event that the batch is accepted by the inspector. Then A occurs if all 10 items of the selected sample belong to the set of 90 non-defectives.

Number of combinations (samples) favorable to A is:

$$\begin{aligned} N(A) &= \binom{90}{10} \\ P(A) &= \frac{N(A)}{N} \\ &= \frac{\binom{90}{10}}{\binom{100}{10}} \qquad = \frac{90!}{10!80!}\,\frac{10!90!}{100!} \\ &\approx (1 - \frac{1}{10})^{10} \approx \frac{1}{e} \qquad \square \end{aligned}$$

Example 5: What is the probability that two cards picked randomly from a full deck are aces?

Solution

$$\begin{aligned}&N = 52 \text{ cards}\\ &n = 4 \quad \text{aces.}\end{aligned}$$

There are $\binom{52}{2}$ equally likely picks.

$N(A) = \binom{4}{2}$ ways are favorable to getting 2 aces.

$$P(A) = \frac{N(A)}{N} = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{6 \cdot 2}{52 \cdot 51} = \frac{6}{26 \cdot 51} = \frac{1}{221} \quad \square$$

Theorem: Given a population of n elements, let $n_1, n_2, ...n_k$ be positive integers such that $n_1 + n_2 + ...n_k = n$. Then there are precisely

$$N = \frac{n!}{n_1! n_2! ... n_k!}$$

ways of partitioning the population into k sub-populations of the prescribed sizes and *order*.

Proof: Order of sub-populations matters.

$$(n_1 = 4, n_2 = 2, n_3, ..., n_k) \neq (n_1 = 2, n_2 = 4, n_3, ..., n_k).$$

Order *within* sub-populations does not matter.

$$
\begin{aligned}
N &= \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-\sum_{i=1}^{k-2} n_i}{n_{k-1}} \\
&= \frac{n!}{n_1!(n-n_1)!}\,\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\,\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} \\
&\quad \cdots \frac{(n-\sum_{i=1}^{k-2} n_i)!}{n_{k-1}!n_k!} \\
&= \frac{n!}{n_1!n_2!\ldots n_k!} \qquad \square
\end{aligned}
$$

Example 6: What is the probability that each of 4 bridge players holds an ace?

$$
\begin{aligned}
n &= 52 \\
n_1 &= n_2 = n_3 = n_4 = 13
\end{aligned}
$$

From the theorem, there are $\frac{n!}{n_1!n_2!n_3!n_4!}$ equally likely deals.

There are $4! = 24$ ways of giving an ace to each player.

Remaining 48 cards can be dealt in $\frac{48!}{12!12!12!12!}$ ways.

Thus these are $24 \cdot \frac{48!}{(12!)^4}$ distinct deals favorable to the desired event.

$$
\begin{aligned}
P(\text{event}) &= 24 \cdot \frac{\frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}} \\
&\approx 0.105
\end{aligned}
$$

Use Stirling's formula to get this approximation.

Stirling's Formula (following Feller)
Let $a_n = \frac{n!}{(n)^n}$ $n = 1, 2, ...$

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \Big/ \frac{n!}{(n)^n} \\
&= \frac{(n+1)n!}{(n+1)^n(n+1)} \, \frac{(n)^n}{n!} \\
&= \frac{1}{\left(1+\frac{1}{n}\right)^n}
\end{aligned}
$$

Let $b_n = n! \left(\frac{e}{n}\right)^n = a_n e^n$

$$
\begin{aligned}
log_e \frac{b_{n+1}}{b_n} &= 1 + log_e \frac{a_{n+1}}{a_n} \\
&= 1 - n \, log_e \, \left(1 + \frac{1}{n}\right) \\
&= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right) \\
&= \frac{1}{2n} - \frac{1}{3n^2} + \cdots
\end{aligned}
$$

Let $\beta_n = n! \, \left(\frac{e}{n}\right)^{n+\frac{1}{2}}$

$$
\begin{aligned}
log_e \frac{\beta_{n+1}}{\beta_n} &= 1 - \left(n + \frac{1}{2}\right) \, log_e \left(1 + \frac{1}{n}\right) \\
&= -\frac{1}{12n^2} + \frac{1}{12n^3} - \cdots < 0.
\end{aligned}
$$

Hence $\beta_{n+1} < \beta_n$. We have shown that β_n is a montone decreasing sequence which is bounded below by 0. Thus $\beta = \lim_{n \to \infty} \beta_n$ exists.

In other words,

$$\beta_n = n! \left(\frac{e}{n}\right)^{n+\frac{1}{2}} \to \beta \qquad \text{a constant}$$

So we can take $n! \sim \beta \cdot (n)^{n+\frac{1}{2}} e^{-(n+1/2)}$. Verify that $\beta = \sqrt{2\pi e}$.

Thus

$$\boxed{n! \sim \sqrt{2\pi}(n)^{n+\frac{1}{2}} e^{-n}} \tag{I}$$

There is a slightly better one.

$$\boxed{n! \sim \sqrt{2\pi}(n)^{n+\frac{1}{2}} e^{-n+\frac{1}{12n}}} \tag{II}$$

Formulas (I) and (II) are respectively the first and second approximations of Stirling.