# ENEE 324H Lecture 6
## Inequalities

Estimating probabilities is necessary where analytic formulas are hard to find. Finding good estimates (upper and lower bounds) is an art. But there are some basic estimates derivable from first principles.

1. **Markov inequality**
   Let $X$ be a non-negative random variable. Let $u$ denote the unit step function

   $$u(x) = \left\{ \begin{array}{ll} 1 & x \geqslant 0 \\ 0 & x < 0 \end{array} \right.$$

   Let $a > 0$. Then it is easy to see that

   $$u(X - a) \leq \frac{X}{a}$$

   Then

   $$E\left(u(X - a)\right) \leq \frac{E(X)}{a}$$

   But

   $$\begin{aligned} E\left(u(X - a)\right) &= 0 \cdot P\{\omega : X(\omega) < a\} + 1 \cdot P\{\omega : X(\omega) \geqslant a\} \\ &= P(X \geqslant a) \end{aligned}$$

   Thus

   $$\boxed{P(X \geqslant a) \leq \frac{E(X)}{a}}$$

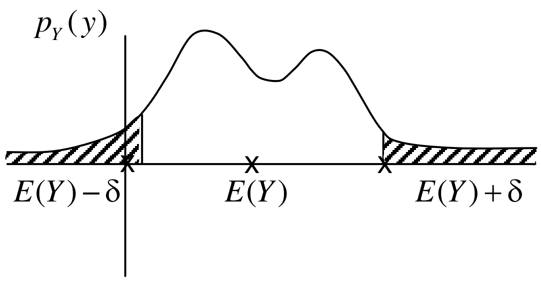
   **Remark** (a) Since $\{\omega : X(\omega) > a\} \subseteq \{\omega : X(\omega) \geqslant a\}$
   it follows that

   $$\begin{aligned} P\{\omega : X(\omega) > a\} &\leq P\{\omega : X(\omega) \geqslant a\} \\ &\leq \frac{E(X)}{a} \end{aligned}$$

**Remark** (b) If the assumption of non-negativity of $X$ is not applicable, one can still write

$$P\{\omega : |X(\omega) - \mu| \geqslant a\} \leq E\left(\frac{|X - \mu|}{a}\right)$$

where $\mu \in \mathbb{R}$ is arbitrary and $a > 0$. This observation leads to the next inequality.




$P(|Y - E(Y)| \geqslant \delta)$

= area under density curve marked by hatch lines

= tail probability

Figure 1: Chebyshev's inequality estimates the tail-probability

2. **Chebyshev inequality**
   Let $Y$ be any real-valued random variable.
   Let $X = \mu + (Y - E(Y))^2$
   Set $a = \delta^2$ for $\delta > 0$.
   Then by Markov's inequality,

$$P((Y - E(Y))^2 \geqslant \delta^2) \leq \frac{E\left((Y - E(Y))^2\right)}{\delta^2}$$

equivalently,

$$\boxed{P(|Y - E(Y)| \geqslant \delta) \leq \frac{Var(Y)}{\delta^2}}$$

3. **Convex functions and Jensen's inequality**
$f : \mathbb{R} \to \mathbb{R}$ is <u>convex</u> if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \text{for } \alpha \in [0,1].$$

From this, it follows that the derivative $f'(x)$ (if it exists) is increasing with $x$, and for any fixed $x_0$, there exists a constant $\lambda$ such that

$$f(x) \geqslant f(x_0) + \lambda(x - x_0)$$

The line with slope $\lambda$, passing through $(x_0, f(x_0))$ is called the supporting line at the point $(x_0, f(_0))$ as in *Figure 2*.

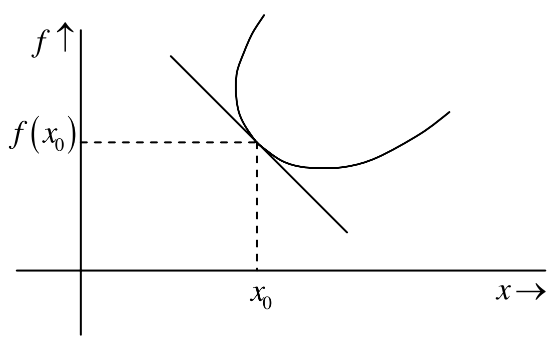


Figure 2

Let $x_0 = E(X)$ for a random variable $X$. Then,

$$\begin{aligned} E(f(X)) &\geqslant E(f(x_0) + \lambda(X - x_0)) \\ &= E(f(E(X))) + E(\lambda(X - E(X)) \\ &= f(E(X)) + \lambda E(X) - \lambda E(X) \\ &= f(E(X)) \end{aligned}$$

Thus, for a convex function $f$,

$$\boxed{E(f(X)) \geqslant f(E(X))}$$

4. **Chernoff's inequality**
Let $X$ be any random variable. Given $\epsilon > 0$, define a new random variable dependent on $X$,

$$Y_\epsilon \;=\; \{ \begin{array}{lll} 1 & if & x \geqslant \epsilon \\ 0 & if & X < \epsilon \end{array}$$

For any $t$, it follows that

$$e^{tX} \geqslant e^{t\epsilon}\, Y_\epsilon$$

Hence

$$\begin{aligned} E(e^{tX}) &\geqslant E(e^{t\epsilon}\, Y_\epsilon) \\ &= e^{t\epsilon}\, E(Y_\epsilon) \\ &= e^{t\epsilon}\, P(X \geqslant \epsilon) \end{aligned}$$

(Here we assume existence of relevant expectations.) Hence,

$$P(X \geqslant \epsilon) \leq e^{-t\epsilon}\, E(e^{tX})$$

The free parameter $t$ in the above inequality can be used to obtain a tighter estimate,

$$\boxed{P(X \geqslant \epsilon) \leq \inf_{t \geqslant 0}\, e^{-t\epsilon}\, E(e^{tX})}$$

Here"infimum" stands for the greatest lower bound.

**Example**: Suppose $X$ is Gaussian with mean 0 and variance 1. Then the density of $X$ is

$$\begin{aligned}
P_X(x) &= \frac{1}{\sqrt{2\pi}} exp\left(\frac{-x^2}{2}\right) \quad -\infty < x < \infty \\
E(e^{tX}) &= \int_{-\infty}^{\infty} e^{tx}\, \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx \\
&= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{1\pi}}\, e^{\left(tx - \frac{x^2}{2} - \frac{t^2}{2}\right)}\, dx \\
&= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{(x-t)^2}{2}}\, dx \\
&= e^{t^2/2}
\end{aligned}$$

Then

$$\begin{aligned}
P(x \geqslant \in) &\leqq \inf_{t \geqslant 0} e^{-t\epsilon + t^2/2} \\
&= e^{-\epsilon^2/2}
\end{aligned}$$