ENEE 420
FALL 2007
COMMUNICATIONS SYSTEMS

ANGLE MODULATION

Throughout, we consider the information-bearing signal $m : \mathbb{R} \to \mathbb{R}$. Its Fourier transform is given by

$$M(f) := \int_{\mathbb{R}} m(t) e^{-j2\pi f t} dt, \quad f \in \mathbb{R}$$

Frequency modulation

The FM waveform $s_{\mathrm{FM}} : \mathbb{R} \to \mathbb{R}$ associated with the the information-bearing signal m is given by

$$s_{\mathrm{FM}}(t) = A_c \cos\left(\theta_{\mathrm{FM}}(t)\right), \quad t \in \mathbb{R}$$

with

$$\theta_{\mathrm{FM}}(t) = 2\pi f_c t + 2\pi k_F \int_0^t m(r) dr, \quad t \in \mathbb{R}$$

Phase modulation

The PM waveform $s_{\mathrm{PM}} : \mathbb{R} \to \mathbb{R}$ associated with the the information-bearing signal m is given by

$$s_{\mathrm{PM}}(t) = A_c \cos\left(\theta_{\mathrm{PM}}(t)\right), \quad t \in \mathbb{R}$$

with

$$\theta_{\mathrm{PM}}(t) = 2\pi f_c t + \pi k_P m(t), \quad t \in \mathbb{R}$$

Single-tone modulating signals

In an attempt to understand how the spectrum of angle-modulated signals is shaped by that of the modulating signal, we consider the simple case of a single-tone modulating signal $m : \mathbb{R} \to \mathbb{R}$, say

$$m(t) = A_m \cos(2\pi f_m t), \quad t \in \mathbb{R}$$

with amplitude $A_m > 0$ and frequency $f_m > 0$. In that case, we note that

$$\begin{aligned} \theta_{\mathrm{FM}}(t) &= 2\pi f_c t + 2\pi k_F \int_0^t A_m \cos(2\pi f_m r)\, dr \\ &= 2\pi f_c t + 2\pi \frac{k_F A_m}{2\pi f_m} \sin(2\pi f_m t) \\ &= 2\pi f_c t + \frac{k_F A_m}{f_m} \sin(2\pi f_m t) \\ &= 2\pi f_c t + \beta \sin(2\pi f_m t) \qquad (1) \end{aligned}$$

where

$$\beta := \frac{\Delta f}{f_m} \quad \text{and} \quad \Delta f := k_F A_m.$$

Next,

$$\begin{aligned} \cos(\theta_{\mathrm{FM}}(t)) &= \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= \Re\left(e^{j2\pi f_c t} e^{j\beta \sin(2\pi f_m t)}\right) \qquad (2) \end{aligned}$$

The function $t \to e^{j\beta \sin(2\pi f_m t)}$ being continuous and periodic with period $T_m = \frac{1}{f_m}$, it admits the Fourier series representation given by

$$e^{j\beta \sin(2\pi f_m t)} = \sum_k c_k e^{j2\pi k f_m t}, \quad t \in \mathbb{R}$$

with

$$c_k = \frac{1}{T_m} \int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi k f_m t} dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

Now fix $k = 0, \pm 1, \pm 2, \ldots$. Upon making the change of variable $x = 2\pi f_m t$, we get

$$\begin{aligned} c_k &= \frac{1}{T_m} \int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi k f_m t} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - kx)} dx \\ &= J_k(\beta) \qquad (3) \end{aligned}$$

where

$$J_k(\beta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin(x)-kx)}dx, \quad \beta \in \mathbb{R}$$

defines the k^{th} order Bessel function of the first kind.

Substituting we find

$$e^{j\beta\sin(2\pi f_m t)} = \sum_k J_k(\beta)e^{j2\pi k f_m t}, \quad t \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} A_c\cos\left(\theta_{\text{FM}}(t)\right) &= A_c\Re\left(e^{j2\pi f_c t}e^{j\beta\sin(2\pi f_m t)}\right) \\ &= A_c\Re\left(e^{j2\pi f_c t}\sum_k J_k(\beta)e^{j2\pi k f_m t}\right) \\ &= A_c\sum_k J_k(\beta)\Re\left(e^{j2\pi f_c t}e^{j2\pi k f_m t}\right) \\ &= A_c\sum_k J_k(\beta)\cos\left(2\pi\left(f_c+kf_m\right)t\right). \end{aligned} \tag{4}$$

In the frequency domain thsi last relationship becomes

$$S_{\text{FM}}(f) = \frac{A_c}{2}\sum_k J_k(\beta)\left(\delta(f-(f_c+kf_m))+\delta(f+(f_c+kf_m))\right). \tag{5}$$

Thus, although the single-tone signal m has frequency content at the frequencies $f = \pm f_m$, the corresponding FM wave has infinite bandwidth since it displays frequency content at the countably infinite set of frequencies

$$f = \pm(f_c + kf_m), \quad k = 0, \pm 1, \ldots.$$

Carson's formula

The realization that the spectrum of s_{FM} has infinite extant leads to the following practical concern: How much bandwidth is needed to transmit s_{FM} without too much distortion?

One answer to this question was given by Carson, and is summarized in the formula that carries his name: Carson's formula states that the transmission bandwidth B_T of the FM wave associated with the single-tone signal m is well approximated by

$$\begin{aligned} B_{T,\text{Carson}} &:= 2f_m + 2\Delta f \\ &= 2f_m\left(1+\beta\right) \end{aligned} \tag{6}$$

since $\Delta f = f_m \beta$ by definition.

One way to generalize Carson's bandwidth formula could proceed *formally* by giving the quantities f_m and β interpretations which do not rely on the specific form of the information-bearing signal m. We do this as follows:

In the single-tone case, the frequency f_m can be interpreted as the cutoff frequency of the signal – In other words, f_m is the bandwidth of the signal. On the other hand, Δf can be viewed as describing the largest possible excursion of the instantaneous frequency from f_c: Indeed, the instantaneous frequency of the FM wave at time t is given by

$$\frac{1}{2\pi}\frac{d}{dt}\theta_{\mathrm{FM}}(t) = f_c + k_F A_m \cos\left(2\pi f_m t\right)$$

and the corresponding deviation in instantaneous frequency at time t is simply

$$\frac{1}{2\pi}\frac{d}{dt}\theta_{\mathrm{FM}}(t) - f_c = k_F A_m \cos\left(2\pi f_m t\right).$$

Therefore, the maximal deviation from f_c is given by

$$\sup\left(\left|k_F A_m \cos\left(2\pi f_m t\right)\right|, \quad t \in \mathbb{R}\right) = k_F A_m = \Delta f.$$

Now consider an information bearing signal which is bandlimited with cutoff frequency $W > 0$. With the discussion for the single-tone modulating signal in mind, it is natural to replace in Carson's formula f_m by W and Δf by

$$D := \sup\left(k_F \left|m(t)\right|, \quad t \in \mathbb{R}\right)$$

This suggests the approximation

$$B_T \simeq B_{T,\mathrm{Carson}}$$

with

$$\begin{aligned} B_{T,\mathrm{Carson}} &:= 2W + 2D \\ &= 2W\left(1+\beta\right) \end{aligned} \tag{7}$$

where β is defined as

$$\beta := \frac{D}{W} = \frac{\sup\left(k_F \left|m(t)\right|, \quad t \in \mathbb{R}\right)}{W}.$$

At this point, you may feel that the generalized Carson's formula discussed above is simply a formal expression without much practical grounding. We now show through an approximation argument (see below) that the bandwidth as given by $B_{T,\text{Carson}}$ is indeed meaningful from an engineering point of view.

The basic idea is to characterize the spectrum of the FM wave associated with a sampled version of the information-bearing signal. Thus, fix $T > 0$. We approximate the information-bearing signal $m : \mathbb{R} \to \mathbb{R}$ by the staircase approximation $m^\star_T : \mathbb{R} \to \mathbb{R}$ given by

$$m^\star_T(t) = m(kT), \quad kT \leq t < (k+1)T$$

with $k = 0, \pm 1, \ldots$. We then replace $\theta_{\text{FM}} : \mathbb{R} \to \mathbb{R}$ as defined above by $\theta^\star_{\text{FM},T} : \mathbb{R} \to \mathbb{R}$ given by

$$\theta^\star_{\text{FM},T}(t) = 2\pi f_c t + 2\pi k_F \int_0^t m^\star_T(r)dr, \quad t \in \mathbb{R}$$

and write

$$s^\star_{\text{FM},T}(t) = A_c \cos\left(\theta^\star_T(t)\right), \quad t \in \mathbb{R}.$$

Note that

$$\begin{aligned} S^\star_{\text{FM},T}(f) &= \int_{\mathbb{R}} A_c \cos\left(\theta^\star_T(t)\right) e^{-j2\pi f t} dt \\ &= A_c \sum\nolimits_k \int_{kT}^{(k+1)T} \cos\left(\theta^\star_T(t)\right) e^{-j2\pi f t} dt. \end{aligned} \tag{8}$$

Now, for $k = 0, 1, \ldots$, with $kT \leq t < (k+1)T$, we have

$$\begin{aligned} \theta^\star_{\text{FM},T}(t) &= 2\pi f_c t + 2\pi k_F \int_0^t m^\star_T(r)dr \\ &= 2\pi f_c t + 2\pi k_F \left(T \sum_{\ell=0}^{k-1} m(\ell T) + m(kT)(t-kT) \right) \\ &= 2\pi (f_c + k_F m(kT))(t-kT) + 2\pi T \left(k f_c + k_F \sum_{\ell=0}^{k-1} m(\ell T) \right) \\ &= 2\pi (f_c + k_F m(kT))(t-kT) + 2\pi \gamma_k T \end{aligned} \tag{9}$$

where we have set

$$\gamma_k := kf_c + k_F \left(\sum_{\ell=0}^{k-1} m(\ell T) \right).$$

Therefore,

$$s^\star_{\mathrm{FM},T}(t) = A_c \cos\left(2\pi(f_c + k_F m(kT))(t - kT) + 2\pi\gamma_k T\right)$$

and direct substitution yields

$$\begin{aligned}
& \int_{kT}^{(k+1)T} \cos\left(\theta^\star_T(t)\right) e^{-j2\pi f t} dt \\
&= A_c \int_{kT}^{(k+1)T} \cos\left(2\pi(f_c + k_F m(kT))(t - kT) + 2\pi\gamma_k T\right) e^{-j2\pi f t} dt \\
&= A_c e^{-j2\pi k f T} \cdot \int_0^T \cos\left(2\pi(f_c + k_F m(kT))\tau + 2\pi\gamma_k T\right) e^{-j2\pi f \tau} d\tau \qquad (10)
\end{aligned}$$

To evaluate this last integral, we note that

$$\begin{aligned}
& \int_0^T e^{\pm j2\pi((f_c + k_F m(kT))\tau + \gamma_k T)} e^{-j2\pi f \tau} d\tau \\
&= e^{\pm j2\pi\gamma_k T} \int_0^T e^{j2\pi(\pm(f_c + k_F m(kT)) - f)\tau} d\tau \\
&= e^{\pm j2\pi\gamma_k T} \cdot \frac{e^{j2\pi(\pm(f_c + k_F m(kT)) - f)T} - 1}{j2\pi\left(\pm(f_c + k_F m(kT)) - f\right)} \\
&= a_k^\pm(f) \frac{\sin\left(\pi\left(\pm(f_c + k_F m(kT)) - f\right)T\right)}{\pi\left(\pm(f_c + k_F m(kT)) - f\right)} \\
&= a_k^\pm(f) \frac{\sin\left(\pi\left(f \mp (f_c + k_F m(kT))\right)T\right)}{\pi\left(f \mp (f_c + k_F m(kT))\right)} \qquad (11)
\end{aligned}$$

with

$$a_k^\pm(f) = e^{j2\pi\delta_k^\pm(f)T}$$

where

$$\delta_k^\pm(f) = \pm\gamma_k + \frac{1}{2}\left(\pm(f_c + k_F m(kT)) - f\right).$$

Now recall that the sinc function $\mathrm{sinc} : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbb{R}.$$

Therefore, for each $k = 0, 1, \ldots$, we have

$$\begin{aligned} & \int_{kT}^{(k+1)T} \cos\left(\theta^\star_T(t)\right) e^{-j2\pi f t} dt \\ = \ & \frac{1}{2} a_k^+(f) \frac{\sin\left(\pi\left(f - (f_c + k_F m(kT))\right) T\right)}{\pi\left(f - (f_c + k_F m(kT))\right)} \\ & + \frac{1}{2} a_k^-(f) \frac{\sin\left(\pi\left(f + (f_c + k_F m(kT))\right) T\right)}{\pi\left(f + (f_c + k_F m(kT))\right)} \\ = \ & \frac{1}{2} a_k^+(f) \cdot \mathrm{sinc}\left(\left(f - (f_c + k_F m(kT))\right) T\right) \\ & + \frac{1}{2} a_k^-(f) \cdot \mathrm{sinc}\left(\left(f + (f_c + k_F m(kT))\right) T\right) \end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned} & \int_0^\infty \cos\left(\theta^\star_T(t)\right) e^{-j2\pi f t} dt \\ = \ & \frac{1}{2} \sum_{k=0}^\infty a_k^+(f) \cdot \mathrm{sinc}\left(\left(f - (f_c + k_F m(kT))\right) T\right) \\ & + \frac{1}{2} \sum_{k=0}^\infty a_k^-(f) \cdot \mathrm{sinc}\left(\left(f + (f_c + k_F m(kT))\right) T\right) \end{aligned} \tag{13}$$

The zeroes of the sinc function occur at $x = \pm\ell$, $\ell = 1, 2, \ldots$, and its main lobe occupies the interval $[-1, 1]$. As a result, for each $k = 0, 1, \ldots$, the main contributions of the terms

$$\frac{1}{2} a_k^\pm(f) \cdot \mathrm{sinc}\left(\left(f \mp (f_c + k_F m(kT))\right) T\right)$$

is taking place on intervals centered at

$$\pm(f_c + k_F m(kT))$$

and of length $2/T$, namely

$$[\pm(f_c + k_F m(kT)) - \frac{1}{T}, \pm(f_c + k_F m(kT)) + \frac{1}{T}$$

Similar arguments could be made for the case $k = -1, -2, \ldots$ and would lead to a similar expression for

$$\int_{-\infty}^0 \cos\left(\theta^\star_T(t)\right) e^{-j2\pi f t} dt, \quad f \in \mathbb{R}.$$

The discussion suggests that most of the spectral content is contained in the interval

$$[\pm(f_c - D) - \frac{1}{T}, \pm(f_c + D) + \frac{1}{T}] \quad k = 0, \pm 1, \ldots$$

since

$$|k_F m(kT)| \leq D, \quad k = 0, \pm 1, \ldots$$

by the definition of D. This leads to estimating the transmission bandwidth of $s^\star_{\mathrm{FM},T}$ as being

$$B_T \simeq 2D + \frac{2}{T}$$

If we sample at the Nyquist rate, then $T = \frac{1}{2W}$, and the information contained in m is recoverable from $m^\star_T$, and the transmission bandwidths of their corresponding FM waveforms should be commensurate. In short,

$$B^\star = 2D + 4W$$

is expected to provide a reasonably good approximation to B_T. Note that

$$B^\star = 2D + 2W + 2W = B_{T,\mathrm{Carson}} + 2W$$

so that this argument provides an approximation to the transmissison bandwith of the FM wave s_{FM} which is more conservative than the one provide by Carson's formula. This can be traced to the fact that the approximation is based on a sampling argument.