ENEE 420
FALL 2007
COMMUNICATIONS SYSTEMS

HOMEWORK # 1: Due September 19, 2007

Please work out the **ten** (10) attached problems. **Show** work and **explain** reasoning. Three (3) problems, selected at random amongst these ten problems, will be graded.

1. ______________________________

Consider a random variable X which takes values 1 or 0 with probability p and $1-p$, respectively, with $0 \leq p \leq 1$. Denote the entropy $H_2(X)$ by $h(p)$.

4.a. Give an expression for $h(p)$.
4.b. Show that $h(p)$ is symmetric about $p = \frac{1}{2}$.
4.c. Compute the derivative $h'(p) = \frac{d}{dp}h(p)$.

2. ______________________________

Assume the existence of two sources, namely $X = (\mathcal{X}, \boldsymbol{p})$ and $Y = (\mathcal{Y}, \boldsymbol{q})$ where $\boldsymbol{p} = (p(x),\ x \in \mathcal{X})$ and $\boldsymbol{q} = (q(y),\ y \in \mathcal{Y})$ are pmfs on the finite sets $\mathcal{X}$ and $\mathcal{Y}$, respectively.

Consider now the combined source $Z = (\mathcal{Z}, \boldsymbol{r})$ where $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ and $\boldsymbol{r} = (r(z),\ z \in \mathcal{Z})$ is the pmf on $\mathcal{Z}$ given by

$$r(z) = p(x)q(y), \quad z = (x, y),\ x \in \mathcal{X},\ y \in \mathcal{Y}.$$

Evaluate the binary entropy $H_2(Z)$ of the source $Z = (\mathcal{Z}, \boldsymbol{r})$ in terms of the binary entropies $H_2(X)$ and $H_2(Y)$ of the component sources $X = (\mathcal{X}, \boldsymbol{p})$ and $Y = (\mathcal{Y}, \boldsymbol{q})$.

3. ______________________________

In Problem **2**, we saw how two sources can be combined into a single source. The present problem attempts to go in the opposite direction: We start with the source $Z = (\mathcal{Z}, \boldsymbol{r})$ where $\mathcal{Z}$ is of the form $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ for some finite sets $\mathcal{X}$ and $\mathcal{Y}$, and $\boldsymbol{r} = (r(z),\ z \in \mathcal{Z})$ is a pmf on $\mathcal{Z}$.

Define the pmfs $\boldsymbol{p} = (p(x),\ x \in \mathcal{X})$ and $\boldsymbol{q} = (q(y),\ y \in \mathcal{Y})$ on the finite sets $\mathcal{X}$ and $\mathcal{Y}$, respectively, by

$$p(x) := \sum_{y \in \mathcal{Y}} r(x, y), \quad x \in \mathcal{X}$$

and

$$q(y) := \sum_{x\in\mathcal{X}} r(x,y), \quad y \in \mathcal{Y}.$$

3.a. The two sources $X = (\mathcal{X}, \boldsymbol{p})$ and $Y = (\mathcal{Y}, \boldsymbol{q})$ are determined by the source $Z = (\mathcal{Z}, \boldsymbol{r})$ but is the converse true? Find a condition under which this is indeed the case.

3.b. Compare the binary entropies $H_2(Z)$, $H_2(X)$ and $H_2(Y)$. Can $H_2(Z)$ be larger than $H_2(X) + H_2(Y)$? Reason intuitively and provide a mathematical proof!

3.c. Explore connections with Problem **2.**

4. ______

A fair coin is tossed in ten independent trials. What is the binary entropy of the sequence of outcomes?

5. ______

Consider a finite set $\mathcal{X}$. Recall that for any set of positive integers $\{\ell(x), x \in \mathcal{X}\}$ satisfying the Kraft Inequality

$$\sum_{x\in\mathcal{X}} 2^{-\ell(x)} \leq 1,$$

there exists a binary prefix code $C : \mathcal{X} \to \mathcal{B}^\star$ such that $\ell_C(x) = \ell(x), x \in \mathcal{X}$.

5.a. Show that for any λ in $(0,1)$, there always exists a prefix code $C_\lambda : \mathcal{X} \to \mathcal{B}^\star$ such that

$$\sum_{x\in\mathcal{X}} 2^{-\ell_{C_\lambda}(x)} \leq \lambda.$$

Is this an interesting fact?

5.b. Is it always true that there exists at least one prefix code $C : \mathcal{X} \to \mathcal{B}^\star$ such that

$$\sum_{x\in\mathcal{X}} 2^{-\ell_C(x)} = 1.$$

Explain your answer!

6. ______

Determine whether there exists a prefix (binary) code with codeword lengths

$$1, 3, 3, 3, 4, 4.$$

In the affirmative, construct such a code.

7. ______

Determine whether there exists a prefix (binary) code with codeword lengths

$$2, 2, 3, 3, 4, 4, 5, 5.$$

In the affirmative, construct such a code.

8.

The prefix (binary) code $C : \mathcal{X} \to \mathcal{B}^\star$ is known to contain the codewords 0, 10 and 110. What is the maximal number of codewords of length 5 that can be contained in C? Determine a lower bound on the corresponding size of $\mathcal{X}$.

9.

Ask your friend to think of an integer number X between 0 and 15. You have to guess the selected number by asking YES/NO questions. What is the minimum number of questions by which you are guaranteed to find the correct answer? You can trivially reach the correct conclusion in no more than 15 questions by simply asking whether the number selected is $0, 1, \ldots, 14$. But can you do better! HINT: Think binary encoding and decision trees!

10.

We have a source, say $X = (\mathcal{X}, \boldsymbol{p})$, and a mapping $g : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{Y}$ is some other finite set, say $\mathcal{Y} = \{g(x),\ x \in \mathcal{X}\}$. Let $Y = (\mathcal{Y}, \boldsymbol{q})$ denote the source induced by this transformation where

$$q(y) := \sum_{x \in \mathcal{X}:\ g(x) = y} p(x), \quad y \in \mathcal{Y}.$$

10.a. Show that $H_2(Y) \leq H_2(X)$ and give an intuitive explanation for this inequality.
10.b. When is it the case that equality holds? Explain.
