Homework 3 Solutions

3.2 The Fourier series representation for any periodic signal $x(t)$ with period T_{per} is given by

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left(j 2\pi \frac{n}{T_{per}} t\right)$$

$$\text{where} \quad x_n = \frac{1}{T_{per}} \int_{-T_{per}/2}^{T_{per}/2} x(t) \exp\left(-j 2\pi \frac{n}{T_{per}} t\right) \quad \text{— (1)}$$

let $\tilde{x}(t)$ denote the "generating pulse", i.e,

$$\tilde{x}(t) = \begin{cases} x(t), & -T_{per}/2 \leq t \leq T_{per}/2 \\ 0, & \text{elsewhere.} \end{cases}$$

$$\text{and} \quad x(t) = \sum_{n=-\infty}^{\infty} \tilde{x}(t - nT_{per})$$

Now,

$$\tilde{X}(f) = \mathcal{F}(\tilde{x}(t)) = \int_{-\infty}^{\infty} \tilde{x}(t) \exp(-j2\pi f t)\, dt = \int_{-T_{per}/2}^{T_{per}/2} x(t) \exp(-j2\pi f t)\, dt \quad \text{— (2)}$$

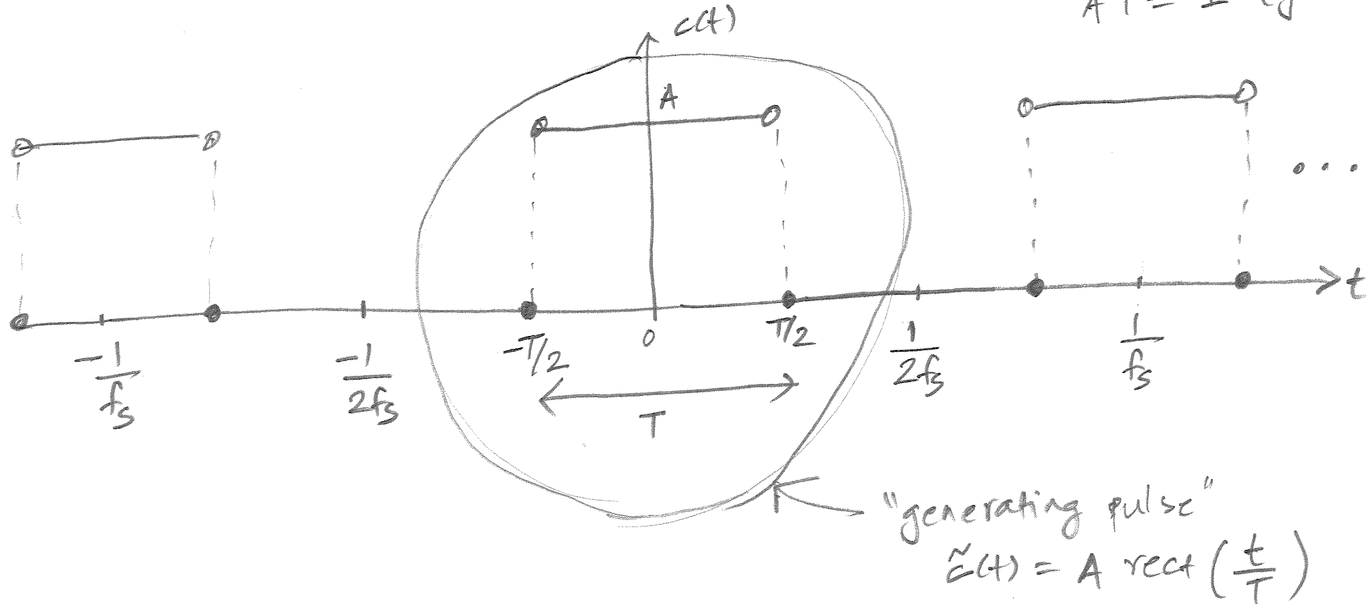
Comparing (1) & (2):

$$x_n = \frac{1}{T_{per}} \tilde{X}\left(\frac{n}{T_{per}}\right).$$

(a) We will now use this to compute the Fourier series representation of the periodic train of rectangular pulses.

In this case, the "generating pulse" is the rectangular pulse of period $T_{per} = \frac{1}{f_s}$ and duration T.

$AT = 1$ (given)



$$\tilde{C}(f) = \mathcal{F}(\tilde{c}(t)) = \underbrace{AT}_{=1} \operatorname{sinc}(fT)$$ (Table A6.3, Pg 764)

$$c_n = f_s \tilde{C}(nf_s) = f_s \operatorname{sinc}(nf_s T)$$

$$c(t) = \sum_{n=-\infty}^{\infty} f_s \operatorname{sinc}(nf_s T) \exp(j2\pi n f_s t)$$

$$s(t) = c(t)\, g(t)$$

$$= \sum_{n=-\infty}^{\infty} f_s \operatorname{sinc}(nf_s T)\, g(t) \exp(j2\pi n f_s t)$$

$$S(f) = \mathcal{F}(s(t)) = \sum_{n=-\infty}^{\infty} f_s \operatorname{sinc}(nf_s T)\, G(f - nf_s)$$

Thus, $S(f)$ consists of frequency-shifted replicas of the original spectrum $G(f)$, with the nth replica being scaled in amplitude by the factor $f_s \operatorname{sinc}(nf_s T)$.

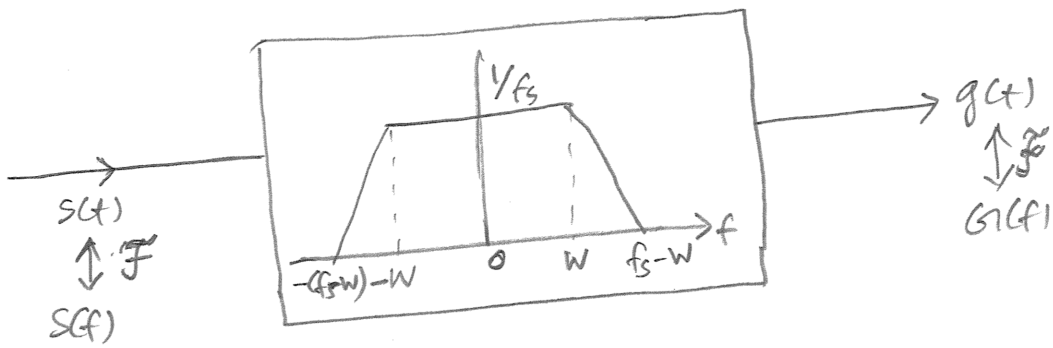
(b) $$S(f) = \underbrace{f_s G(f)}_{\uparrow \text{ recover!}} + f_s \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathrm{sinc}(n f_s T)\, G(f - n f_s)$$

In accordance with the Sampling Theorem, assume

(i) $g(t)$ is band-limited, i.e. $G(f) = 0$ for $|f| \geq W$

(ii) sampling frequency $f_s > 2W$.

Then, the different frequency-shifted replicas of $G(f)$ in the construction of $S(f)$ will <u>not</u> overlap. Therefore, $g(t)$ can be exactly recovered by using a low-pass filter as shown.



3.3 (a) $g(t) = \mathrm{sinc}(200t)$

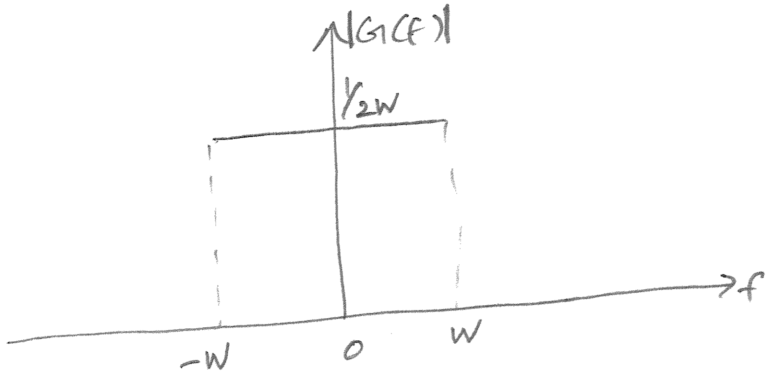
Know $\mathrm{sinc}(2Wt) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$

(Table A6.3, Pg 764

or use

$\mathrm{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\mathrm{sinc}(fT)$

together with <u>duality property</u> of Fourier transform



$\therefore$ $g(t) = \mathrm{sinc}(200t)$ has bandwidth $W = 100\,\text{Hz}$

$\Rightarrow$ Nyquist rate $= 200$ samples/sec ; Nyquist interval $= \frac{1}{200}$ seconds.

(b) $g(t) = \text{sinc}^2(200t)$

$\Rightarrow G(f) = \mathcal{F}(\text{sinc}(200t)) \overset{\text{convolution}}{*} \mathcal{F}(\text{sinc}(200t))$

$\Rightarrow$ Bandwidth is <u>twice</u> that of $\text{sinc}(200t)$, i.e.,

$$BW = 200\,Hz$$

$\therefore$ Nyquist rate $=$ 400 samples/second

Nyquist interval $= \frac{1}{400}$ seconds.

(c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

Since the Fourier Transform is linear, bandwidth is determined by the highest frequency component of $\text{sinc}(200t)$ or $\text{sinc}^2(200t)$.

$\Rightarrow BW = 200\,Hz$

$\therefore$ Nyquist rate $=$ 400 samples/second

Nyquist interval $= \frac{1}{400}$ seconds.

3.4 (a) $s(t) = m_\delta(t) * h(t)$

where $m_\delta(t) = \sum_{n=-\infty}^{\infty} m(n)\,\delta(t-n)$ $\quad (T_s = 1\,s)$

$$S(f) = M_\delta(f)\,H(f)$$

$$H(f) = 0.45\,\text{sinc}(0.45f)\,e^{-j\pi 0.45f} \qquad (T = 0.45\,s)$$

$$M(f) = \mathcal{F}(m(t)) = \mathcal{F}(A_m \cos(2\pi\, 0.25\, t))$$

$$= \frac{A_m}{2}\left[\delta(f-0.25) + \delta(f+0.25)\right]$$

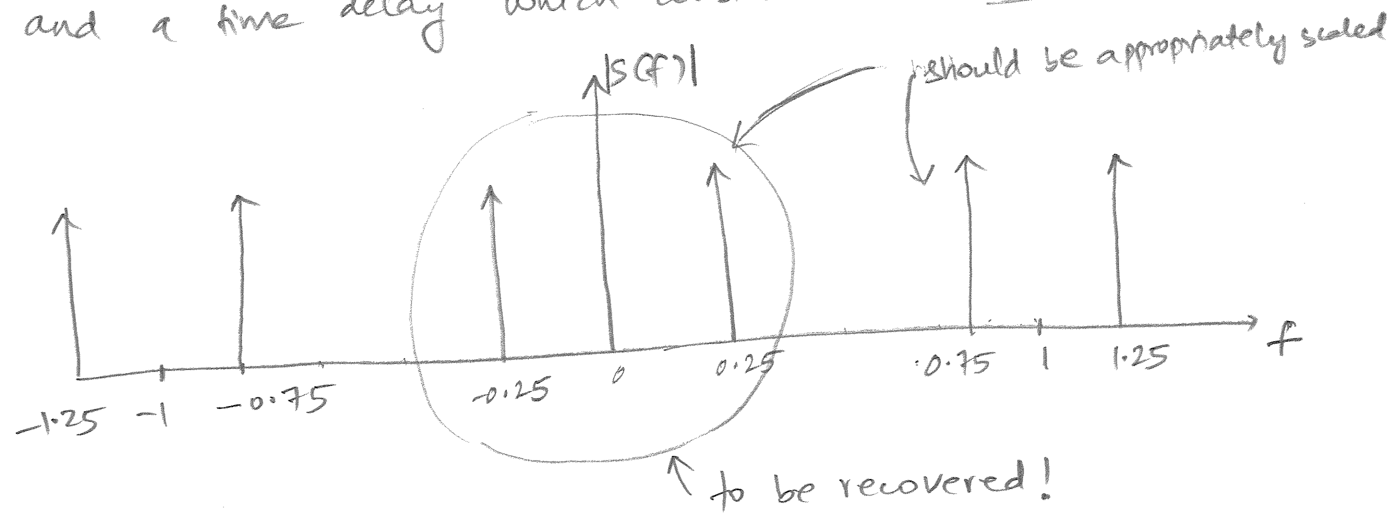
$$M_\delta(f) = \sum_{k=-\infty}^{\infty} M(f-k) \qquad (f_s = 1\ \mathrm{Hz})$$

$$= \sum_{k=-\infty}^{\infty} \frac{A_m}{2}\left[\delta(f-0.25-k) + \delta(f+0.25-k)\right]$$

$$S(f) = \frac{A_m}{2}\, 0.45\, \mathrm{sinc}(0.45f)\, e^{-j\pi 0.45 f} \sum_{k=-\infty}^{\infty}\left[\delta(f-0.25-k) + \delta(f+0.25-k)\right]$$

(b) $S(f)$ contains two δ-functions centered around the origin (at -0.25 and $+0.25$) and this pattern is shifted and repeated every 1 Hz.

There is also a scaling factor with the sinc function and a time delay which constitute the <u>aperture effect</u>.



Using a low-pass filter, the centre two deltas alone are recovered. The scaling factor may then be cancelled out to give the original signal (with some <u>delay</u>).

3.5 sampling frequency $f_s = 1\,KHz$

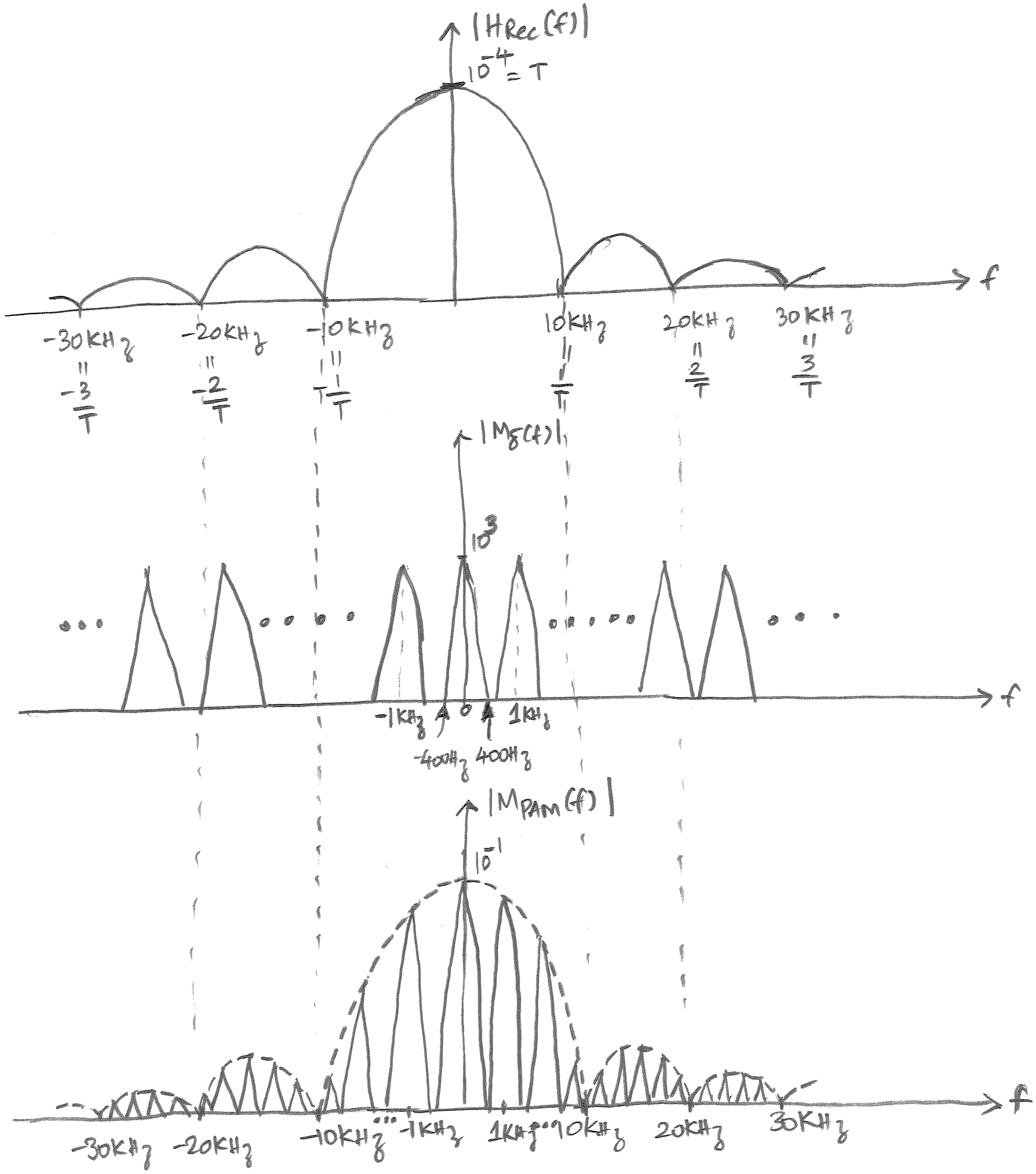
Pulse duration $T = 0.1\,ms$

$$M_{PAM}(f) = M_\delta(f)\, H_{Rec}(f)$$

where

$$M_\delta(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s)$$

$$H_{Rec}(f) = T e^{-j\pi f T} \operatorname{sinc}(fT)$$



3.6 The magnitude response of the equalizer is given by:

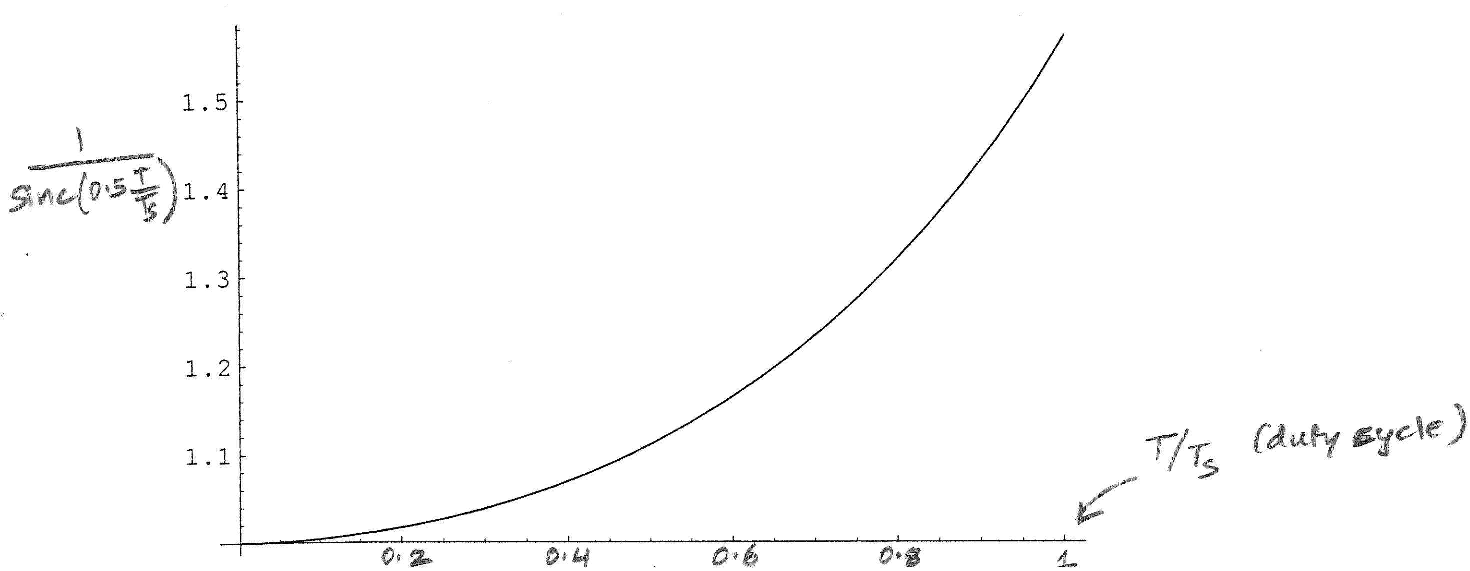
$$\frac{1}{|H_{Rec}(f)|} = \frac{1}{T\,\mathrm{sinc}(fT)} = \frac{\pi f}{\sin(\pi f T)}$$

At the receiver, since an **LPF** is normally used to filter out the frequency components above the highest frequency of the message signal, the maximum equalization will indeed correspond to this highest frequency. Given the highest frequency as $f = f_s/2$, max. equalization normalized to that at zero frequency (i.e. by $1/T$) is

$$\frac{1}{\mathrm{sinc}(0.5 f_s T)} = \frac{(\pi/2)(T/T_s)}{\sin((\pi/2)(T/T_s))}.$$

T/T_s is infact the duty cycle of the sampling pulses. A plot is shown below. It can be seen that for $T/T_s \le 0.1$, equalization needed is negligible and may be omitted altogether. Indeed, for $T/T_s = 0.1$,

$$\frac{1}{\mathrm{sinc}\left(0.5\frac{T}{T_s}\right)} = 1.0041.$$



3.16 Correction: The question should read "A sinusoidal signal has a total duration . . ." .

For a sinusoidal signal,

$$10 \log_{10} (SNR)_0 = 1.8 + 6R \overset{?}{=} 40 \text{ dB}$$

$$R = \left\lceil \frac{40 - 1.8}{6} \right\rceil = 7 \text{ bits/sample}$$

$$\text{Bit rate} = 8000 \text{ samples/second} \times 7 \text{ bits/sample}$$

$$= 56 \text{ Kbps}.$$

$\therefore$ storage needed for 10s duration $= 56 \times 10$ Kbits

$= 560$ Kbits.

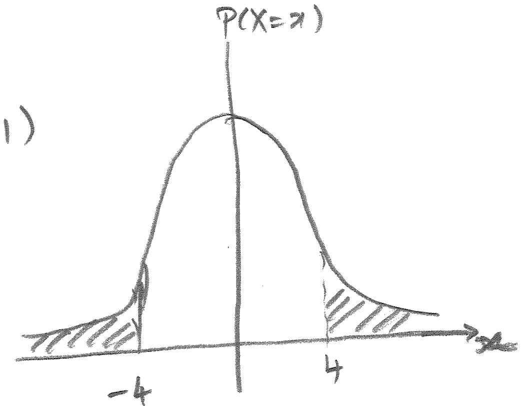
3.17 (a) $X \sim \mathcal{N}(0,1)$

i.e. Gaussian with mean $= 0$, variance $= 1$

$$\Pr\{|X| \geq 4\} = 1 - 2\,\underbrace{\Phi(-4)}_{\text{cum. dist. fnc. of } \mathcal{N}(0,1)}$$

$$= \cancel{1} - \left[\cancel{1} + \text{erf}\left(\frac{-4}{\sqrt{2}}\right)\right]$$

$$= \underline{0.999937}$$



where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad , x \in \mathbb{R}$

and we used $\Phi(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right], \quad x \in \mathbb{R}$.

(erf called the "error function" can be computed from tables or using software like Matlab).

(b) For a uniform quantizer of mid tread type,

$$\Delta = \frac{2A}{L}$$

Assuming quantizer input is in $[-4, +4] \Rightarrow A = 4$.

For Δ small enough, the quantization error is uniform $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$.

$$\therefore \text{Var}[E] = \frac{\Delta^2}{12}.$$

With R bits per sample, $L = 2^R$. Hence, output SNR

$$(SNR)_0 = \frac{P}{\text{Var}[E]} = \frac{P}{\Delta^2/12}$$

$$= \frac{12\,P}{4A^2 \cdot 2^{-2R}} = \frac{3P}{16} \cdot 2^{2R}$$

where P = Avg. power in message $= \mathbb{E}[X^2]$
$= \text{Var}[X] = 1$

$$\therefore (SNR)_0 = \frac{3}{16} 2^{2R}$$

i.e. $(SNR)_0\big|_{dB} = 10 \log_{10} (SNR)_0 = 6R - 7.2 \text{ dB}.$

Comparing with Example 3.1, where $(SNR)_0 = 6R + 1.8$ dB, we see that the output SNR is higher for a sinusoidal signal input than for a Gaussian-distributed random variable.

3.18

$R = 7$ bits / sample

Bit rate $= 50 \times 10^6$ bits/s

(a) $f_s = \frac{50 \times 10^6}{7}$ samples /second

$\therefore$ max. message bandwidth $= \frac{f_s}{2} = 3.57 \times 10^6$ Hz.

(b) For full-load sinusoidal modulating signal, from Example 3.1 (Pg 197)

$$10 \log_{10}(SNR)_0 = 1.8 + 6R$$
$$= 43.8 \text{ dB}.$$

<u>3.19</u> Since Δ_i is small, the quantization error can be assumed to be uniformly distributed in $\left(-\frac{\Delta_i}{2}, \frac{\Delta_i}{2}\right)$ when the message falls in the ith interval. Thus,

$$\Pr\{E(M)=e \mid M\in I_i\} = \begin{cases} \frac{1}{\Delta_i}, & -\frac{\Delta_i}{2} \le e \le \frac{\Delta_i}{2} \\ 0, & \text{otherwise} \end{cases}$$

Also $\Pr(M\in I_i) = p_i$ (given).

Now, mean-square value of quantization error

$$\mathbb{E}[E(M)^2] = \int_{-\infty}^{\infty} e^2 \Pr\{E(M)=e\}\, de$$

$$= \int_{-\infty}^{\infty} e^2 \left(\sum_i \Pr(M\in I_i) \Pr\{E(M)=e \mid M\in I_i\}\right) de$$

$$= \int_{-\infty}^{\infty} e^2 \left(\sum_i p_i \Pr\{E(M)=e \mid M\in I_i\}\right) de$$

$$= \sum_i p_i \int_{-\infty}^{\infty} e^2 \cdot \Pr\{E(M)=e \mid M\in I_i\}\, de$$

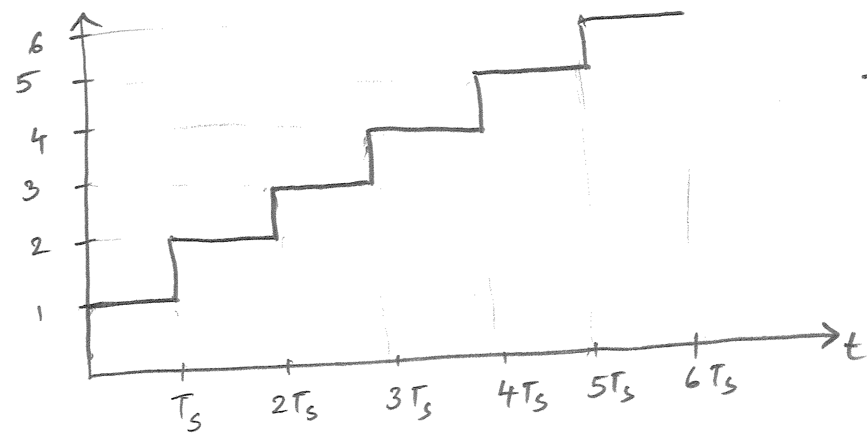
$$= \sum_i p_i \int_{-\Delta_i/2}^{\Delta_i/2} e^2 \cdot \frac{1}{\Delta_i}\, de$$

$$= \sum_i p_i \cdot \frac{1}{\Delta_i} \left.\frac{e^3}{3}\right|_{-\Delta_i/2}^{\Delta_i/2} = \underline{\underline{\frac{1}{12}\sum_i \Delta_i^2 \cdot p_i}}$$

3.22 Codewords:

$$\underbrace{0\,0\,1}_{1},\ \underbrace{0\,1\,0}_{2}\ \ \underbrace{0\,1\,1}_{3}\ \ \underbrace{1\,0\,0}_{4}\ \ \underbrace{1\,0\,1}_{5}\ \ \underbrace{1\,1\,0}_{6}$$

$\therefore$ sampled version of analog signal is:



$T_s = 3\,T_b$