Homework 4 Solutions

3.25

$m(t) = A \tanh(\beta t)$

To avoid slope overload, need

$$\frac{\Delta}{T_s} \geqslant \max\left|\frac{d\,m(t)}{dt}\right|$$

$$\frac{d\,m(t)}{dt} = A\beta\,\mathrm{sech}^2(\beta t)$$

$$= A\beta \cdot \frac{1}{\cosh^2(\beta t)}$$

$$= A\beta \cdot \left(\frac{2}{e^{\beta t} + e^{-\beta t}}\right)^2$$

$\therefore$ $\frac{d\,m(t)}{dt}$ is max. when $t = 0$ and

$$\max\left|\frac{d\,m(t)}{dt}\right| = A\beta$$

$\therefore$ need $\Delta \geqslant A\beta T_s$

3.26 $m(t) = A_m \sin 2\pi f_m t$

applied to delta mod. with step-size Δ.

(a) When does slope-overload distortion occur?

$$\frac{\Delta}{T_s} < \max\left|\frac{dm(t)}{dt}\right|$$

$$\frac{dm(t)}{dt} = \left(A_m \cos 2\pi f_m t\right) 2\pi f_m$$

$$\max\left|\frac{dm(t)}{dt}\right| = A_m \cdot 2\pi f_m$$

$$\frac{\Delta}{T_s} < A_m \cdot 2\pi f_m \quad \text{i.e.} \quad A_m > \frac{\Delta}{2\pi f_m T_s}$$

(b) max. power that can be transmitted without slope-overload distortion?

$$\left.\begin{array}{l}\text{Power for sinusoidal signal with} \\ \text{amplitude } A_m\end{array}\right\} = A_m^2/2$$

For no slope-overload distortion:

$$A_m \leq \frac{\Delta}{2\pi f_m T_s}$$

$$\therefore \text{ max. power} = \frac{\Delta^2}{8\pi^2 f_m^2 T_s^2}$$

3.27

$W = 3.4\,KHz$

$f_s = 10 \times f_{Nyquist} = 10 \times 6.8\,KHz = 68\,KHz.$

$\Delta = 100\,mV.$

1 KHz sinusoidal signal.

max A_m ? (to avoid slope overload)

$$A_m \leq \frac{\Delta}{2\pi f_m T_s} = \frac{100 \times 10^{-3} \times 68 \times \cancel{10^3}}{2\pi \times \cancel{10^3}} = \underline{1.08\ V}$$

3.28 avg. signal-to-(quantization) noise ratio.

Power spectral density of the granular noise $\Big\} = S_N(f) = \dfrac{\Delta^2}{3f_s}$

Note: Correction in denominator

(a) Sinusoidal signal : A, f_m

Δ chosen to avoid slope overload.

Avg quantization noise power ?

$$N = \frac{\Delta^2}{3f_s} \times 2W$$

$$\Delta = A\,2\pi f_m T_s \qquad \text{(from Prob. 3.26)}$$

$$\therefore N = \frac{A^2 4\pi^2 f_m^2 \times 2W}{3f_s^3}$$

$$= \frac{8\pi^2 A^2 f_m^2 W}{3f_s^3}$$

(b) $$\text{SNR} = \frac{A^2/2}{8\pi^2 A^2 f_m^2 W / 3f_s^3} = \frac{3f_s^3}{16\pi^2 f_m^2 W}$$

3.29 $W = 5\,KHz$

sinusoidal signal: $A_M = 1\,V$, $f_m = 1\,kHz$.

$f_s = 50\,kHz$

(a) min. Δ to avoid slope overload?

From Prob. 3.26, need

$$\Delta \geq \frac{A_m 2\pi f_m}{f_s}$$

$$= 0.126\ V$$

(b) SNR?

From Prob 3.28,

$$SNR = \frac{3 f_s^3}{16\pi^2 f_m^2 W} = \frac{3 \times 50^3 \times \cancel{10^9}}{16\pi^2 \times \cancel{10^6} \times 5 \times \cancel{10^3}} = 475.42$$

$$(SNR)_{dB} = 10\log_{10}(475.42) = 26.77\ dB$$

3.30 $W = 3\ KHz$, $\Delta = 0.1V$

$f_s = 10 \times f_{Nyquist} = 60 KHz$

(a) sinusoidal signal: $f_m = 1 KHz$.

Max A_m for no slope overload?

$$A_m \le \frac{\Delta}{2\pi f_m T_s} \quad \text{(from Prob. 3.26)}$$

$$= \frac{0.1 \times 60 \times 10^3}{2\pi \times 10^3} = 0.96\ V$$

(b) (i) Pre-filtered:

Here the entire (quantization) noise power affects the signal. So, calculate the variance of (quantization) noise.

Assume quantization noise is uniformly distributed in $[-\Delta, +\Delta]$.

$$\sigma_Q^2 = \int_{-\Delta}^{\Delta} \frac{1}{2\Delta} q^2\, dq = \frac{\Delta^2}{3}$$

$$\therefore (SNR)_{pre\text{-}filtered} = \frac{A_m^2/2}{\Delta^2/3} = \frac{3 \times 0.96^2}{2 \times 0.1^2} = 138.24$$

$$(SNR)_{pre\text{-}filtered, dB} = 10 \log_{10}(138.24) = 21.41 dB.$$

(ii) Post-filtered:

In this case, the (quantization) noise power in the frequency range $[-W, W]$ only will affect the signal.

From Prob 3.28,

$$(SNR)_{post\,filtered} = \frac{3 f_s^3}{16 \pi^2 f_m^2 W}$$

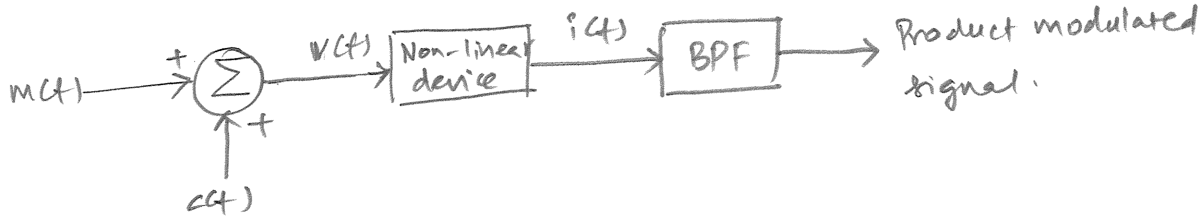
$$= \frac{\cancel{3} \times 60^3 \times \cancel{10^9}}{16 \pi^2 \times \cancel{10^6} \times \cancel{3} \times \cancel{10^3}}$$

$$= 1369.22$$

$$(SNR)_{post\,filtered,\,dB} = 10 \log_{10}(1369.22) = 31.36 \text{ dB}$$

2.1 (a) Let $c(t) = A_c \cos \pi f_c t$

i.e. a sinusoidal signal at frequency $f_c/2$.



$$v(t) = m(t) + A_c \cos \pi f_c t$$

$$\begin{aligned}
i(t) &= a_1 v(t) + a_3 v^3(t) \\
&= a_1 (m(t) + A_c \cos \pi f_c t) + a_3 (m(t) + A_c \cos \pi f_c t)^3 \\
&= a_1 (m(t) + A_c \cos \pi f_c t) \\
&\quad + a_3 \left(m^3(t) + 3 A_c m^2(t) \cos \pi f_c t + 3 A_c^2 m(t) \cos^2 \pi f_c t + A_c^3 \cos^3 \pi f_c t \right)
\end{aligned}$$

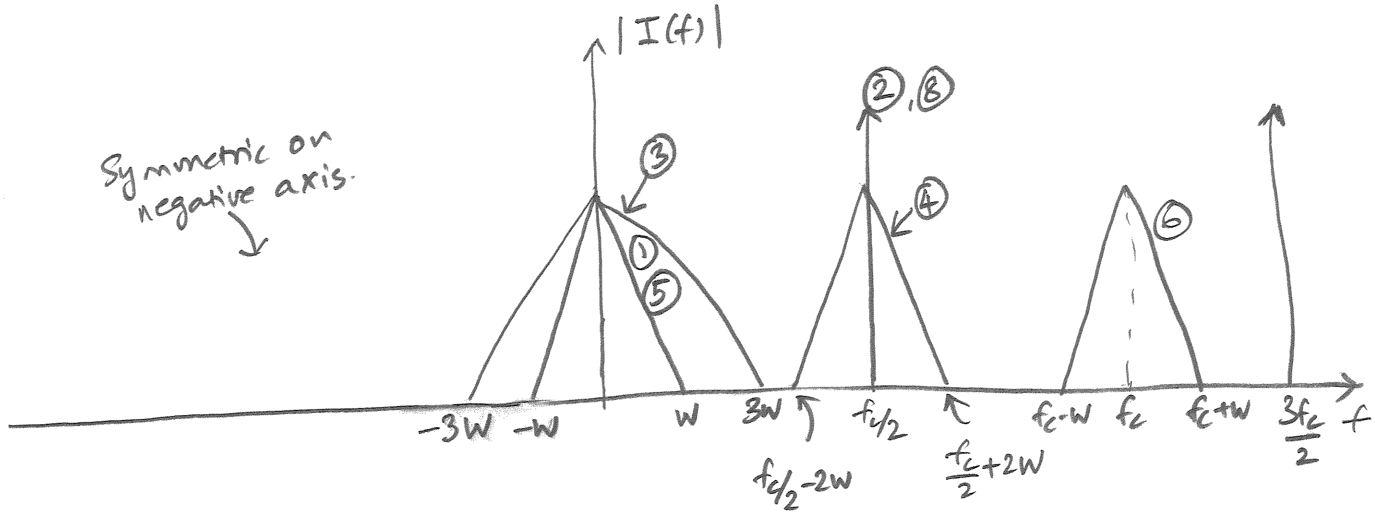
$$\cos^2 \pi f_c t = \frac{1 + \cos 2\pi f_c t}{2}$$

$$\begin{aligned}
\cos^3 \pi f_c t &= \left(\frac{1 + \cos 2\pi f_c t}{2} \right) \cos \pi f_c t \\
&= \frac{\cos \pi f_c t}{2} + \frac{\cos 3\pi f_c t + \cos \pi f_c t}{4} \\
&= \frac{\cos 3\pi f_c t + 3 \cos \pi f_c t}{4}
\end{aligned}$$

product modulated signal

$$\begin{aligned}
i(t) = {} & a_1 (\overset{①}{m(t)} + \overset{②}{A_c \cos \pi f_c t}) \\
& + a_3 \Big(\overset{③}{m^3(t)} + \overset{④}{3 A_c m^2(t) \cos \pi f_c t} + \overset{⑤}{\frac{3 A_c^2}{2} m(t)} + \underbrace{\overset{⑥}{\frac{3 A_c^2}{2} m(t) \cos 2\pi f_c t}}_{} \\
& + \underset{⑦}{\frac{A_c^3}{4} \cos 3\pi f_c t} + \underset{⑧}{\frac{3 A_c^3}{4} \cos \pi f_c t} \Big)
\end{aligned}$$

Assume $m(t)$ is band limited to $-W \le f \le W$. Then,



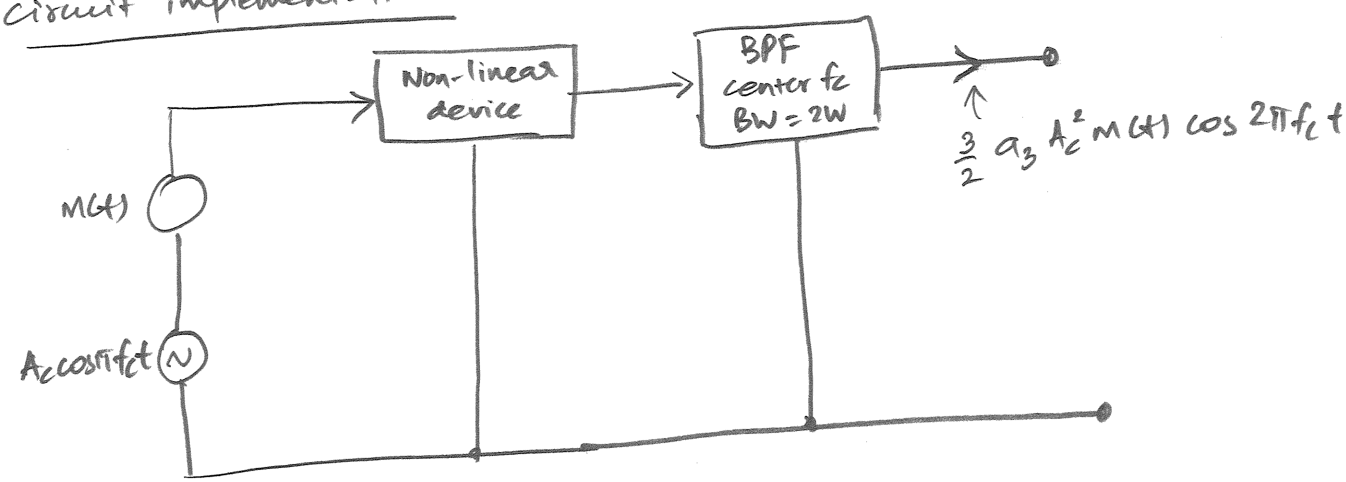
Take BPF centered at f_c with bandwidth $= 2W$.

Output of BPF will be term 6 only if

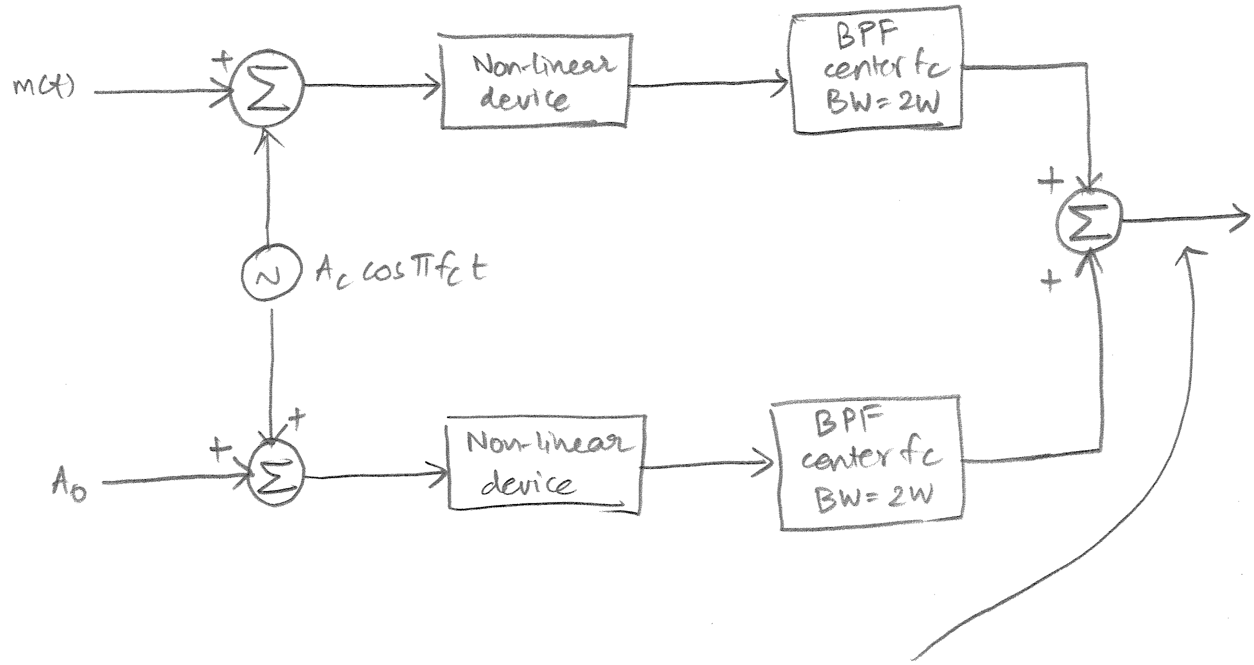
$$\left.\begin{array}{ll} f_c - W > \frac{f_c}{2} + 2W & \Leftrightarrow f_c > 6W \\ \text{and} \quad f_c + W < \frac{3f_c}{2} & \Leftrightarrow f_c > 2W \end{array}\right\} \Rightarrow \text{Need } \underline{f_c > 6W}$$

$$\boxed{\text{Output of BPF} = \frac{3}{2} a_3 A_c^2 m(t) \cos 2\pi f_c t}$$ ← Product modulated signal

Circuit implementation:



(b) To obtain an amplitude modulated signal, we need an additional sinusoidal signal added to the product modulated signal from part (a). This is achieved by using two branches as shown below. The first branch is same as the product modulator. The second branch has a constant input instead of a message signal.



$$\left.\begin{matrix}\text{Amplitude modulated} \\ \text{signal}\end{matrix}\right\} = \frac{3}{2} a_3 A_c^2 \left(A_0 + m(t)\right) \cos 2\pi f_c t$$

$$= \frac{3}{2} a_3 A_c^2 A_0 \left(1 + \frac{1}{A_0} m(t)\right) \cos 2\pi f_c t$$

Amplitude sensitivity is controlled by the input A_0.

2.6 Square-law detector

(a) $V_1(t) = A_c(1 + k_a m(t)) \cos 2\pi f_c t$

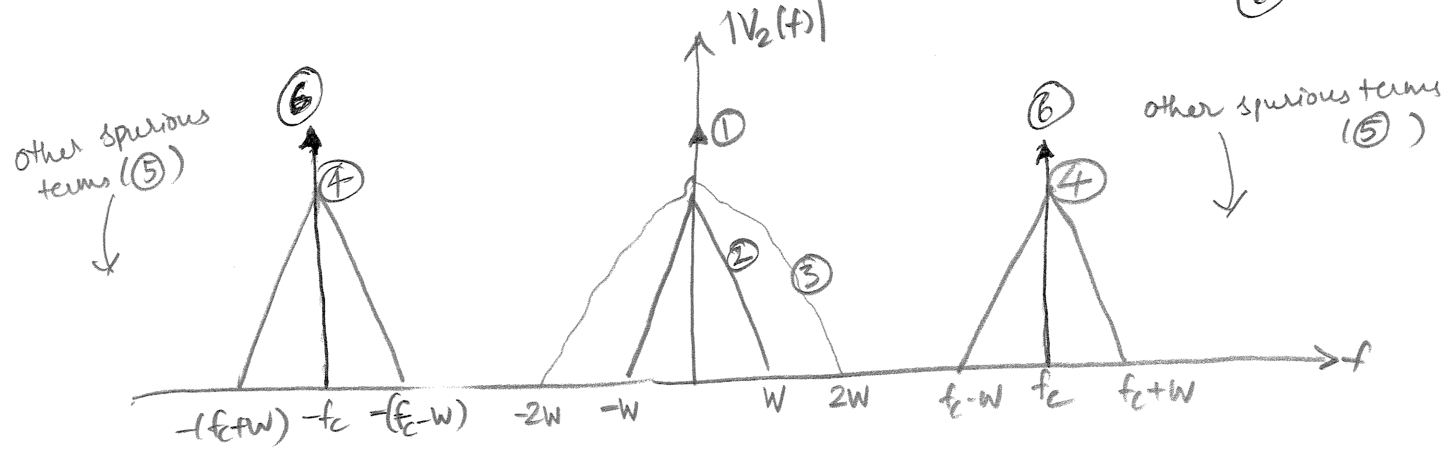
$$V_2(t) = a_1 V_1(t) + a_2 V_1^2(t)$$

$$= a_1 A_c(1 + k_a m(t)) \cos 2\pi f_c t$$

$$+ a_2 A_c^2 \left(1 + k_a^2 m^2(t) + 2k_a m(t)\right)\left(\frac{1 + \cos 4\pi f_c t}{2}\right)$$

$$V_2(t) = \overset{(1)}{\frac{a_2 A_c^2}{2}} + \overset{(2)}{a_2 A_c^2 k_a m(t)} + \overset{(3)}{\frac{a_2 A_c^2}{2} k_a^2 m^2(t)} + \overset{(4)}{a_1 A_c k_a m(t) \cos 2\pi f_c t}$$

$$+ \underset{(5)}{\frac{a_2 A_c^2}{2}\left(1 + k_a^2 m^2(t) + 2 k_a m(t)\right)\cos 4\pi f_c t} + \underset{(6)}{a_1 A_c \cos 2\pi f_c t}$$



(b) Pass $V_2(t)$ through a LPF of BW = W Hz

Need: $f_c - W > W \iff \underline{\underline{f_c > 2W}}$

$$\text{Output of LPF} = \underbrace{\frac{a_2 A_c^2}{2}}_{\substack{\text{can be subtracted}\\ \text{out as all parameters}\\ \text{are known at receiver.}}} + \underbrace{a_2 A_c^2 k_a}_{\substack{\text{scaling factors}\\ \text{can also be}\\ \text{removed.}}} m(t) + \text{"}\underbrace{\frac{a_2 A_c^2}{2} k_a^2 m^2(t)}_{\substack{BW = 2W\\ \text{so, portion of it gets through.}}}\text{"}$$

$$\text{ratio of } \frac{\text{wanted signal}}{\text{unwanted signal}} = \frac{a_2 A_c^2 k_a m(t)}{a_2 \frac{A_c^2}{2} k_a^2 m^2(t)} = \frac{2}{k_a m(t)}$$

If $|k_a m(t)| << 1$, then desired ratio is high.

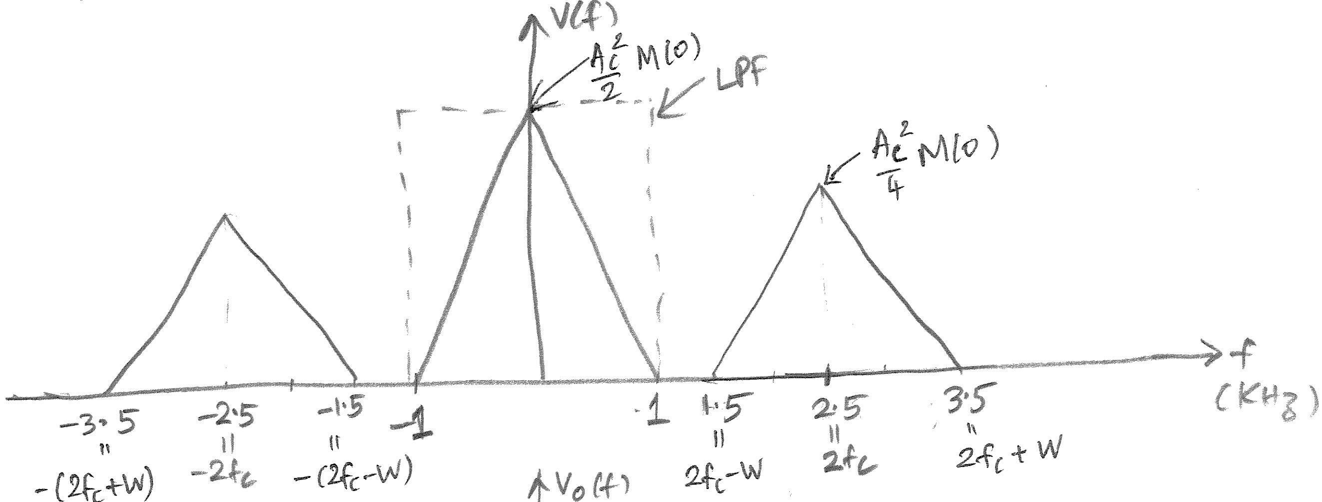
2.8 $v(t)$ = output of product modulator in coherent detector

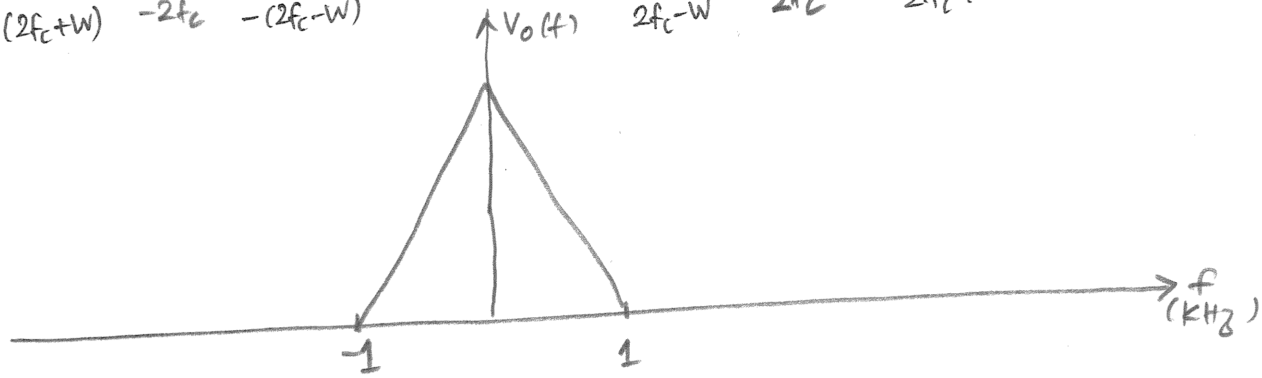
$v_0(t)$ = final output of coherent detector.

(Ref. Fig 2.7, P95; Fig 2.8, P96)

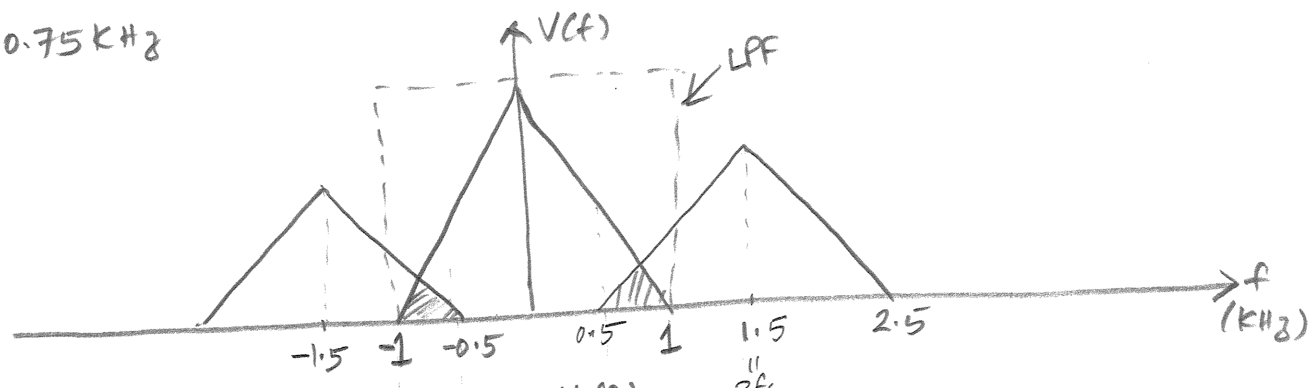
$W = 1\ KHz$.

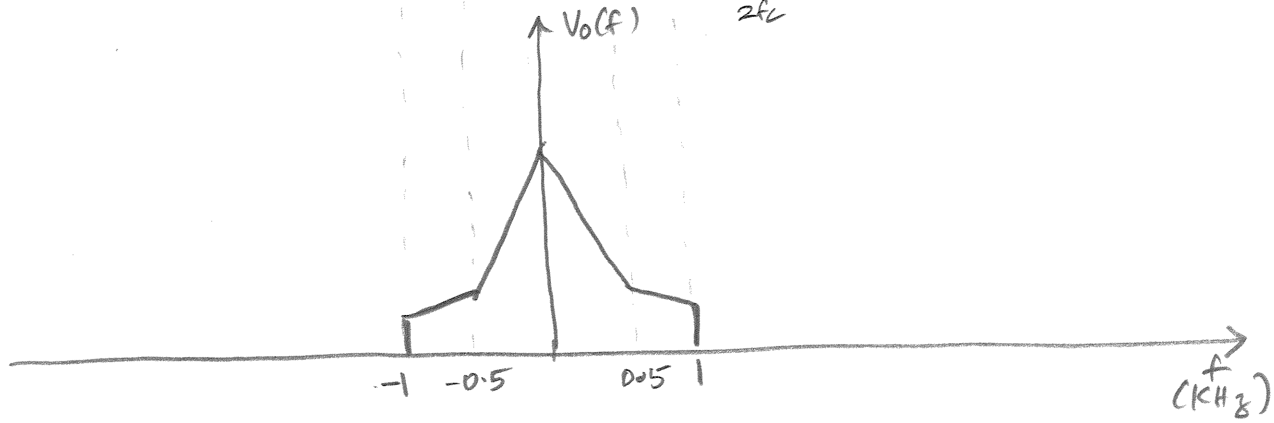
(a) $f_c = 1.25\ KHz$





(b) $f_c = 0.75\ KHz$

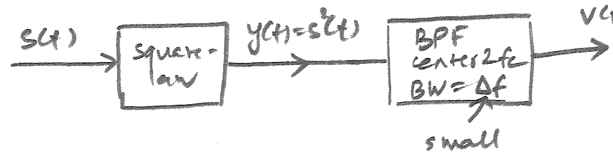




- To avoid overlap, need $2f_c - W > W$, i.e., $\underline{f_c > W}$.

- Lowest carrier frequency for each component of modulated signal to be uniquely determined by $m(t)$ $\Big\}$ $= \underline{1\text{ KHz}}$

2.11 (a) $s(t) = A_c \cos(2\pi f_c t)\, m(t)$



$$y(t) = s^2(t)$$

$$= A_c^2 \cos^2(2\pi f_c t)\, m^2(t)$$

$$= \frac{A_c^2}{2} m^2(t)\,(1 + \cos 4\pi f_c t)$$

$$Y(f) = \frac{A_c^2}{2}\left(M(f) * M(f)\right) + \frac{A_c^2}{4} M(f) * \underbrace{\left(M(f) * \left[\delta(f-2f_c) + \delta(f+2f_c)\right]\right)}_{M(f-2f_c) + M(f+2f_c)}$$

$$= \frac{A_c^2}{2} \int_{-\infty}^{\infty} M(\lambda)\, M(f-\lambda)\, d\lambda$$

$$+ \frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} M(\lambda)\, M(f-2f_c-\lambda)\, d\lambda + \int_{-\infty}^{\infty} M(\lambda)\, M(f+2f_c-\lambda)\, d\lambda\right]$$

(b) At $f = 2f_c$,

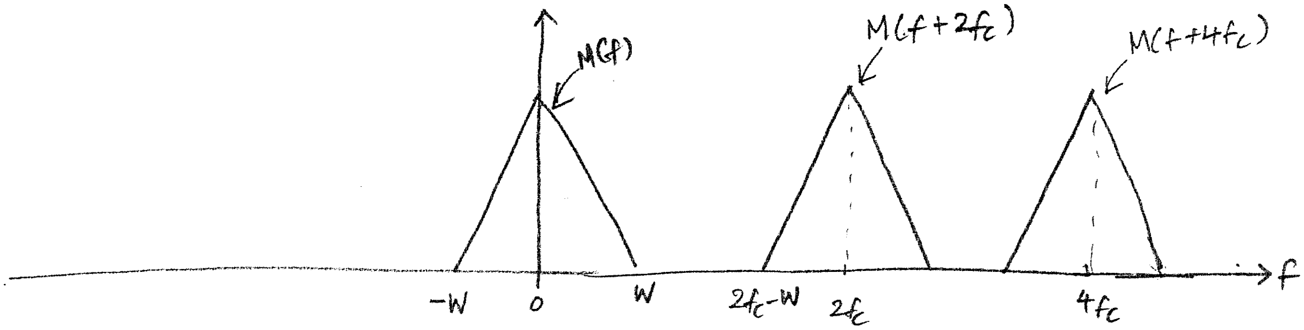
$$Y(2f_c) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)\, M(2f_c-\lambda)\, d\lambda$$

$$+ \frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} M(\lambda)\, \underbrace{M(-\lambda)}_{M^*(\lambda)}\, d\lambda + \int_{-\infty}^{\infty} M(\lambda)\, M(4f_c-\lambda)\, d\lambda\right]$$

$$= \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)\, M(2f_c-\lambda)\, d\lambda$$

$$+ \frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} |M(\lambda)|^2\, d\lambda + \int_{-\infty}^{\infty} M(\lambda)\, M(4f_c-\lambda)\, d\lambda\right] \quad \text{—} (*)$$

If $m(t)$ is band limited to $-W \le f \le W$, then:



For $2f_c - W > W$ i.e. $f_c > W$, there is no overlap between $M(f)$ and $M(f+2f_c)$ or $M(f+4f_c)$. Hence, the first and last integrals in (*) are zero.

$$Y(2f_c) = \frac{A_c^2}{4} \int_{-\infty}^{\infty} |M(\lambda)|^2 \, d\lambda = \frac{A_c^2}{4} E$$

where E = energy of $m(t)$

$$= \int_{-\infty}^{\infty} |M(\lambda)|^2 \, d\lambda .$$

Similarly $Y(-2f_c) = \frac{A_c^2}{4} E$

Since the pass band Δf of the BPF is small, the output in the frequency domain is approximately:

$$V(f) \cong \frac{A_c^2}{4} E \, \Delta f \left[\delta(f - 2f_c) + \delta(f + 2f_c) \right]$$

Hence,

$$v(t) = \mathcal{F}^{-1}(V(f)) \cong \frac{A_c^2}{2} E \, \Delta f \cos(4\pi f_c t) .$$