ENEE 420
Fall 2007

Test #2
Answer Key

#1. Note that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \theta \in \mathbb{R}$$

so that

$$\cos^2\theta = \cos 2\theta + \sin^2\theta = \cos 2\theta + 1 - \cos^2\theta$$

∴

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \theta \in \mathbb{R}.$$

Therefore,

$$\begin{aligned}\cos^3\theta &= \cos^2\theta \cdot \cos\theta \\ &= \frac{\cos\theta + \cos\theta\cos 2\theta}{2} \\ &= \frac{\cos\theta + (\cos(\theta+2\theta) + \cos(\theta-2\theta))/2}{2} \\ &= \frac{1}{4}(\cos 3\theta + 3\cos\theta), \quad \theta \in \mathbb{R}.\end{aligned}$$

If $y(t) = A_c m(t)\cos^3(2\pi f_c t)$, then

$$y(t) = \frac{A_c}{4} m(t)\left(\cos(3 \cdot 2\pi f_c t) + 3\cos(2\pi f_c t)\right), \quad t \in \mathbb{R}$$

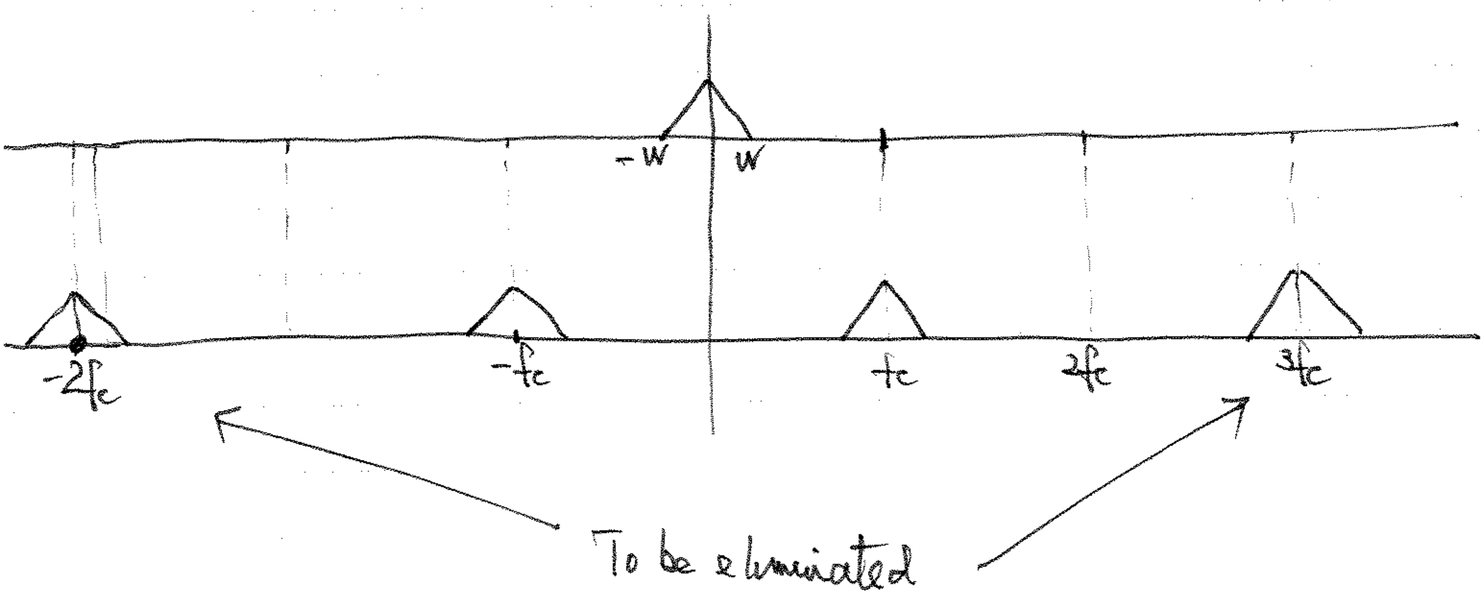
so that

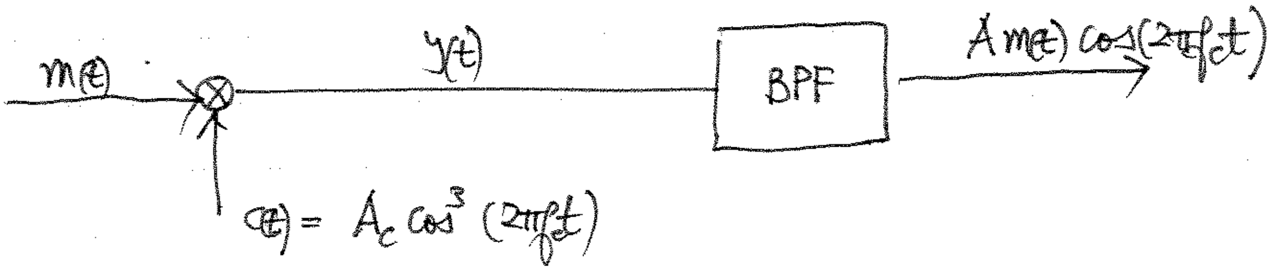
$$Y(f) = \frac{3A_c}{8}\left(M(f+f_c) + M(f-f_c)\right) + \frac{A_c}{8}\left(M(f+3f_c) + M(f-3f_c)\right)$$

$$f \in \mathbb{R}$$

#b

- Pictorially, with $m \in LP(W)$ where $W < f_c$,



- 

where A and A_c are related by $A = \frac{3A_c}{4}$, and the BP filter $\in BP(f_c, W)$ (i.e., its passband is $|\pm f_c - f| \le W$).

#c If $c(t) = \cos^2(2\pi f_c t)$, then

$$y(t) = A_c\, m(t) \cos^2(2\pi f_c t)$$

$$= \frac{A_c}{2} m(t)\left(1 + \cos(4\pi f_c t)\right), \quad t \in \mathbb{R}$$

- so that

$$Y(f) = \frac{A_c}{4}\left(M(f+f_c) + M(f-f_c)\right) + \frac{A_c}{4}\left(M(f+2f_c) + M(f-2f_c)\right), \quad f \in \mathbb{R}$$

∴ This signal y has no frequency content at $f = \pm f_c$, so that no amount of linear filtering (applied to y) can result in a signal that would have frequency content at $f = \pm f_c$ as would be the case for $s(t) = A\, m(t) \cos(2\pi f_c t)$ with $m \in LP(W)$ -

#2. Note that

$$m(t) = 10 \cos(1000\pi t) + 5 \cos(1500\pi t)$$
$$= 10 \cos(2\pi(500)t) + 5 \cos(2\pi(750)t)$$
$$= 10 \cos(2\pi(2f_c)t) + 5 \cos(2\pi(3f_c)t), \quad t \in \mathbb{R}$$

with $f_c = 250$.

a/ The signal m is bandlimited, with $m \in LP(W)$ and

$$W = 3f_c = 750 \quad (Hz)$$

so *at minimum*, we should take the sampling rate $f_s = \frac{1}{T_s}$ to be at least the Nyquist rate of m, namely

$$f_{Nyq} = 2W$$
$$= 6f_c$$
$$= 1{,}500 \text{ Hz}$$

Of course, with DM, the requirement is $f_s \gg f_{Nyq} = 1{,}500$, or equivalently

$$T_s \ll (1{,}500)^{-1} \text{ sec}$$

b/ Slope overload is avoided if

$$\frac{\Delta}{T_s} \gg \max\left\{ \left|\frac{dm(t)}{dt}\right| ,\ t \in I \right\}$$

where I is the interval where the signal is observed, e.g., say $I = \mathbb{R}$ for simplicity (As will be seen shortly, the specific interval plays no role).

But

$$\frac{dm(t)}{dt} = \frac{d}{dt}\left[10\cos(2\pi(2f_c)t) + 5\cos(2\pi(3f_c)t) \right]$$

$$= 2\pi\Big[(-10 \times 2f_c)\sin(2\pi(2f_c)t) + (-5 \times 3f_c)\sin(2\pi(3f_c)t) \Big]$$

so that

$$\left|\frac{dm(t)}{dt}\right| = 2\pi \left| (10 \times 2f_c)\sin(2\pi(2f_c)t) + (5 \times 3f_c)\sin(2\pi(3f_c)t) \right|$$

The maximum of $\left|\frac{dm(t)}{dt}\right|$ is achieved at either the maximum or minimum of the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) = (10 \times 2f_c)\sin(2\pi(2f_c)t) + (5 \times 3f_c)\sin(2\pi(3f_c)t)$$

One could in principle compute $\frac{d}{dt} g(t)$ and solve (for t) the equation $\frac{d}{dt} g(t) = 0$. This approach, though correct, is quite tedious as it does not yield a "neat" sol'n. Instead, since $|\sin\theta| \le 1$ for all $\theta \in \mathbb{R}$, we see that

$$\left| \frac{dm(t)}{dt} \right| \le 2\pi \left[(10 \times 2f_c) + (5 \times 3 f_c) \right], \quad t \in \mathbb{R}$$

so that

$$\left| \frac{dm(t)}{dt} \right| \le 70\pi f_c, \quad t \in \mathbb{R}$$

The earlier requirement on Δ and T_s is now implied by

$$\frac{\Delta}{T_s} \ge 70\pi f_c$$

i.e.,

$$\frac{\Delta}{T_s} \ge 70 \times \pi \times 250 = 17{,}500\pi.$$

#3

a/ It is easy to check graphically that the needed condition

$$H_{VSB}(f+f_c) + H_{VSB}(f-f_c) = 1, \quad |f| \leq W$$

b/ Here

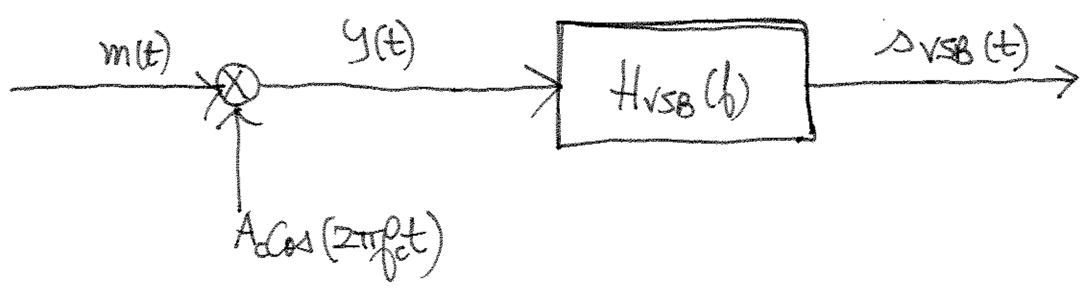
$$B_T = W + \frac{W}{2} = \frac{3W}{2}$$

Note that

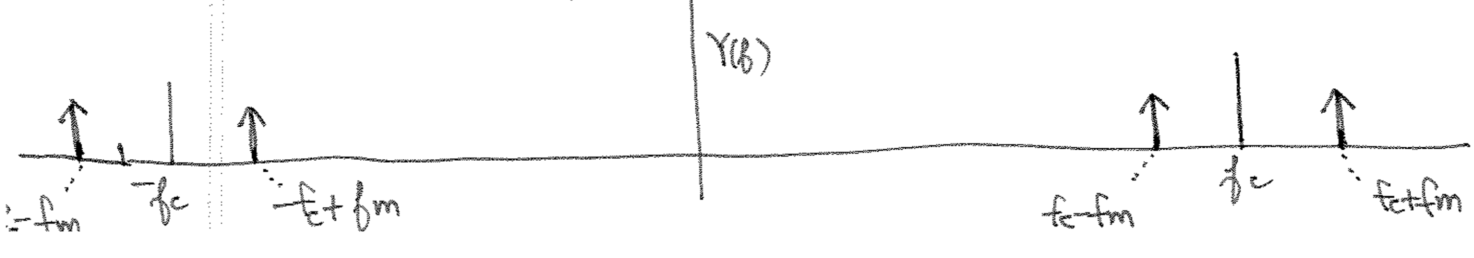
$$y(t) = m(t) A_c \cos(2\pi f_c t)$$

$$= A_c A_m \cos(2\pi f_m t) \cos(2\pi f_c t)$$

$$= \frac{A_c A_m}{2} \left[\cos(2\pi(f_m + f_c)t) + \cos(2\pi(f_c - f_m)t) \right]$$



Since $f_m = cW$, $0<c<1$, and $W < f_c$, we have $f_m < f_c$ - Pictorially



c) With $\frac{1}{2}<c<1$, $f_c+f_m = f_c + cW > f_c + \frac{1}{2}W$
$< f_c + W$

and

$$f_c - f_m = f_c - cW < f_c - \frac{W}{2}$$

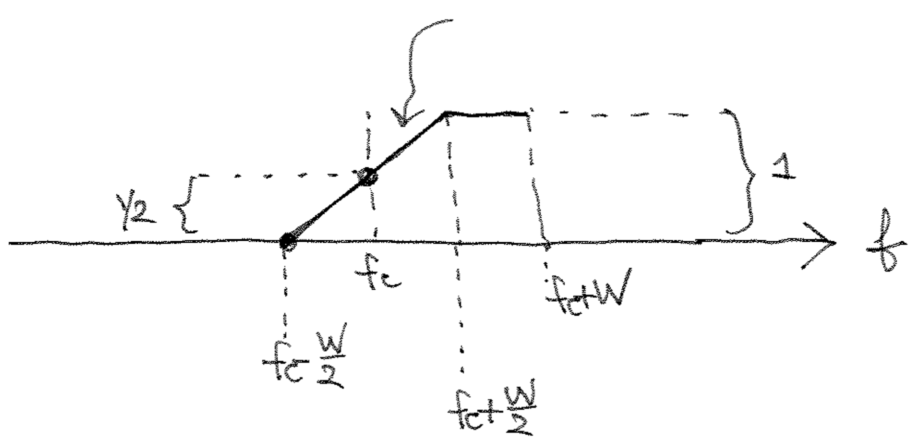
Therefore, direct inspection of $Y(f)$ and $H_{VSB}(f)$ shows that

$$s_{VSB}(t) = \frac{A_c A_m}{2} \cos\left(2\pi(f_c+f_m)t\right), \quad t \in \mathbb{R}.$$

d) With $0<c<\frac{1}{2}$,

$$f_c - \frac{W}{2} < f_c - f_m < f_c + f_m < f_c + \frac{W}{2}.$$

Also,



Straight line on $|f_c - f| \leq \frac{W}{2}$:

$$H_{VSB}(f) = \frac{1}{W}\left(f - \left(f_c - \frac{W}{2}\right)\right) + 0$$

$$= \frac{1}{W}\left(f - f_c + \frac{W}{2}\right)$$

At $f = f_c - f_m = f_c - cW$,

$$H_{VSB}(f) = \frac{1}{W}\left(f_c - cW - f_c + \frac{W}{2} \right) = \left(\frac{1}{2} - c\right).$$

At $f = f_c + f_m = f_c + cW$,

$$H_{VSB}(f) = \frac{1}{W}\left(f_c + cW - f_c + \frac{W}{2} \right) = \frac{1}{2} + c$$

Direction inspection of $Y(f)$ and $H_{VSB}(f)$ now yields

$$s_{VSB}(t) = \frac{A_c A_m}{2}\left[\left(\frac{1}{2}+c\right)\cos(2\pi(f_c+f_m)t) + \left(\frac{1}{2}-c\right)\cos(2\pi(f_c-f_m)t)\right]$$

#

#4.

1/ We have

$$s(t) = A\left[\sin(2\pi(f_c+f_a)t) - \sin(2\pi(f_c-f_a)t)\right] + B\cos(2\pi f_c t)$$

$$= A\left[\sin(2\pi f_c t)\cos(2\pi f_a t) + \sin(2\pi f_a t)\cos(2\pi f_c t) - \sin(2\pi f_c t)\cos(2\pi f_a t) + \sin(2\pi f_a t)\cos(2\pi f_c t)\right] + B\cos(2\pi f_c t)$$

$$= \left[2A\sin(2\pi f_a t) + B\right]\cos(2\pi f_c t).$$

$$= \left[B + m(t)\right]\cos(2\pi f_c t).$$

You can also write $s(t) = A_c\left(1 + \frac{2A}{B}\sin(2\pi f_a t)\right)\cos(2\pi f_c t)$ with $A_c = B$ and $m(t) = \frac{2A}{B}\sin(2\pi f_a t)$

$\therefore$ AM with $m(t) = 2A\sin(2\pi f_a t)$.

2/ We have

$$s(t) = \left(2A\sin(2\pi f_a t) + B\right)\cos(2\pi f_c t) = s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)$$

$\therefore$ $s_I(t) = 2A\sin(2\pi f_a t) + B$ and $s_Q(t) = 0$, $t \in \mathbb{R}$

$\therefore$ Both s_I and s_Q are LP.

3/ It is now clear that $a(t) = \sqrt{s_I(t)^2 + s_Q(t)^2} = \left|2A\sin(2\pi f_a t) + B\right|$

The envelope detector (applied to s) yields a. Thus, perfect reconstruction is possible if $2A\sin(2\pi f_a t) + B \geq 0$, $t \in \mathbb{R}$ which happens if $-2A + B \geq 0$ $\|$ $\boxed{2A \leq B}$