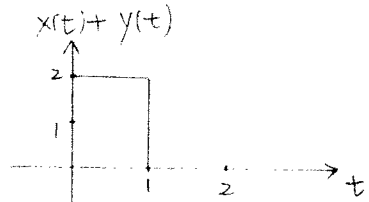
# HW#1 Solutions

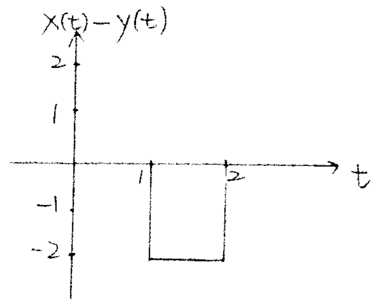
1. $\boxed{2.1\text{-}2}$ (a)

Figure P2.1-2(a) :

$$E_x = \int_0^2 1^2\,dt = 2\ ,\quad E_y = \int_0^1 1^2 dt + \int_1^2 (-1)^2 dt = 1+1 = 2$$




$$E_{x+y} = \int_0^1 2^2 dt = 4$$



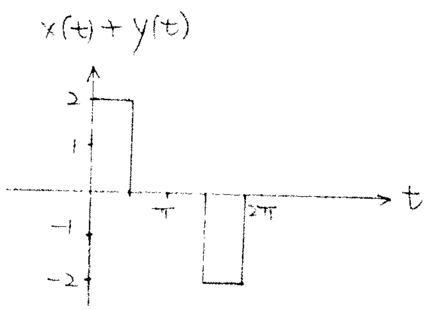

$$E_{x-y} = \int_1^2 (-2)^2 dt = 4$$

$$\Rightarrow\ E_{x+y} = E_{x-y} = E_x + E_y = 4$$

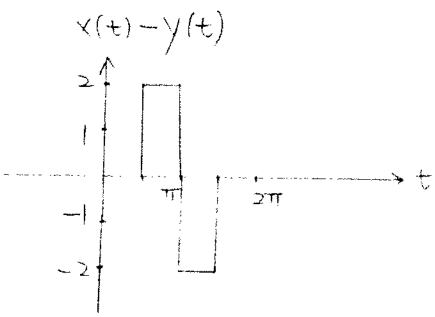
Figure P2.1-2(b) :

$$E_x = \int_0^{\pi} 1^2 dt + \int_{\pi}^{2\pi} (-1)^2 dt = \pi + \pi = 2\pi$$

$$E_y = \int_0^{\frac{\pi}{2}} 1^2 dt + \int_{\frac{\pi}{2}}^{\pi} (-1)^2 dt + \int_{\pi}^{\frac{3\pi}{2}} 1^2 dt + \int_{\frac{3\pi}{2}}^{2\pi} (-1)^2 dt = \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = 2\pi$$




$$E_{x+y} = \int_0^{\frac{\pi}{2}} 2^2 dt + \int_{\frac{3\pi}{2}}^{2\pi} (-2)^2 dt = 4\times\frac{\pi}{2} + 4\times\frac{\pi}{2} = 4\pi$$




$$E_{x-y} = \int_{\frac{\pi}{2}}^{\pi} 2^2 dt + \int_{\pi}^{\frac{3\pi}{2}} (-2)^2 dt = 4\times\frac{\pi}{2} + 4\times\frac{\pi}{2} = 4\pi$$

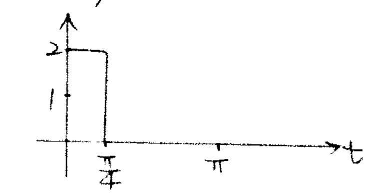
$$\Rightarrow\ E_{x+y} = E_{x-y} = E_x + E_y = 4\pi$$

1. $\boxed{2.1-2}$ (b)

Figure P2.1-2(c):

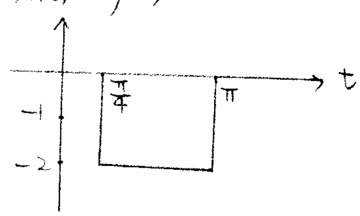
$$\left.\begin{aligned} E_x &= \int_0^{\frac{\pi}{4}} 1^2\,dt + \int_{\frac{\pi}{4}}^{\pi} (-1)^2\,dt = \frac{\pi}{4} + \frac{3}{4}\pi = \pi \\ E_y &= \int_0^{\pi} 1^2\,dt = \pi \end{aligned}\right\} \Rightarrow E_x + E_y = \pi + \pi = 2\pi$$

$x(t) + y(t)$




$$E_{x+y} = \int_0^{\frac{\pi}{4}} 2^2\,dt = 4 \times \frac{\pi}{4} = \pi$$

$x(t) - y(t)$




$$E_{x-y} = \int_{\frac{\pi}{4}}^{\pi} (-2)^2\,dt = 4 \times \frac{3}{4}\pi = 3\pi$$

$$\Rightarrow \quad E_{x+y} \neq E_x + E_y$$

$$E_{x-y} \neq E_x + E_y$$

2. $\boxed{2.1-3}$

$$T_0 \triangleq \frac{2\pi}{\omega_0}$$

$$\begin{aligned} \frac{1}{T_0}\int_0^{T_0} C^2 \cos^2(\omega_0 t + \theta)\,dt &= \frac{\omega_0}{2\pi} C^2 \int_0^{\frac{2\pi}{\omega_0}} \cos^2(\omega_0 t + \theta)\,dt \\ &= \frac{\omega_0}{2\pi} C^2 \left[ \int_0^{\frac{2\pi}{\omega_0}} \frac{1}{2}\,dt + \int_0^{\frac{2\pi}{\omega_0}} \frac{1}{2}\cos(2\omega_0 t + 2\theta)\,dt \right] \\ &= \frac{\omega_0}{2\pi} C^2 \left[ \frac{1}{2} \cdot \frac{2\pi}{\omega_0} + \left( \frac{1}{\omega_0} \sin(2\omega_0 t + 2\theta) \Big|_0^{\frac{2\pi}{\omega_0}} \right) \right] \\ &= \frac{\omega_0}{2\pi} C^2 \left[ \frac{\pi}{\omega_0} + \frac{1}{\omega_0}\big( \sin(4\pi + 2\theta) - \sin(2\theta) \big) \right] \\ &= \frac{\omega_0}{2\pi} C^2 \left[ \frac{\pi}{\omega_0} + \frac{1}{\omega_0} \cdot 0 \right] \\ &= \frac{\omega_0}{2\pi} C^2 \frac{\pi}{\omega_0} \\ &= \frac{C^2}{2} \end{aligned}$$

3. $\boxed{2.2\text{-}1}$

(1) When $a$ is real:

$$\int_{-\infty}^{\infty}(e^{-at})^2 dt = \int_{-\infty}^{\infty} e^{-2at} dt = \begin{cases} \int_{-\infty}^{\infty} 1\, dt = \infty, & \text{if } a=0 \\ \frac{1}{-2a}\left[e^{-2at}\Big|_{-\infty}^{\infty}\right] = \infty, & \text{if } a\neq 0 \end{cases}$$

$$\frac{1}{T}\int_{-T/2}^{T/2}(e^{-at})^2 dt = \begin{cases} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 1\, dt = \frac{T}{T} = 1 & \text{, if } a=0 \\ \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2at} dt = \frac{1}{T}\frac{1}{2a}\left(e^{aT}-e^{-aT}\right) & \text{, if } a\neq 0 \end{cases}$$

$$\Rightarrow \lim_{T\to\infty}\int_{-T/2}^{T/2}(e^{-at})^2 dt = \begin{cases} \lim_{T\to\infty} 1 = 1 & \text{, if } a=0 \\ \lim_{T\to\infty}\frac{1}{2aT}\left(e^{aT}-e^{-aT}\right) = \infty & \text{, if } a\neq 0 \end{cases}$$

$\therefore$ when $a$ is real and $a\neq 0$, $e^{-at}$ is neither an energy nor a power signal
when $a=0$, $e^{-at}$ is not an energy signal, but a power signal.

(2) When $a$ is imaginary:

Let $a=kj$ $\quad \int_{-\infty}^{\infty}|e^{-at}|^2 dt = \int_{-\infty}^{\infty}|e^{-kjt}|^2 dt = \int_{-\infty}^{\infty} 1\, dt = \infty$

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|e^{-at}|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} 1\, dt = \lim_{T\to\infty}\frac{T}{T} = \lim_{T\to\infty} 1 = 1$$

$\therefore$ when $a$ is imaginary, $e^{-at}$ is a power signal with power $P_g = 1$

3. $\boxed{2.2\text{-}2}$ Let $c=\alpha+j\beta$, $\alpha\neq 0$.

$$\int_{-\infty}^{\infty}|e^{-ct}|^2 dt = \int_{-\infty}^{\infty}|e^{-\alpha t}\cdot e^{-j\beta t}|^2 dt = \int_{-\infty}^{\infty}|e^{-\alpha t}|^2\cdot|e^{-j\beta t}|^2 dt = \int_{-\infty}^{\infty} e^{-2\alpha t} dt \underset{\boxed{2.2\text{-}1}(a)}{=} \infty$$

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|e^{-ct}|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} e^{-2\alpha t} dt = \lim_{T\to\infty}\frac{1}{2\alpha T}\left(e^{\alpha T}-e^{-\alpha T}\right) = \infty$$

$\boxed{2.2\text{-}1}$ (a)

$\therefore$ $e^{-ct}$ is neither an energy signal nor a power signal

4. $\boxed{2.3\text{-}4}$

$g(t)$ is a periodic signal with period $T$ and average power $P_g$.

$\Rightarrow g(at)$ is a periodic signal with period $\frac{T}{|a|}$ (assume $a \neq 0$, $a$ is real)

average power of $g(at)$ is $\displaystyle\lim_{\frac{T}{|a|}\to\infty} \frac{|a|}{T}\int_{-\frac{T}{2|a|}}^{\frac{T}{2|a|}} |g(at)|^2\,dt$ — let $\tau = at$

$$= \lim_{\frac{T}{|a|}\to\infty} \frac{|a|}{aT}\int_{-\frac{aT}{2|a|}}^{\frac{aT}{2|a|}} |g(\tau)|^2\,d\tau$$

$$= \begin{cases} \displaystyle\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |g(\tau)|^2\,d\tau = P_g, & a>0 \\[2ex] \displaystyle\lim_{T\to\infty}\frac{-1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} |g(\tau)|^2\,d\tau = P_g, & a<0 \end{cases}$$

$g(at)$ is a periodic signal with period $\frac{T}{|a|}$ and average power $P_g$.

5. $\boxed{2.4\text{-}1}$

(a) $\dfrac{\tan(t)}{2t^2+1}\delta(t) = \dfrac{\tan(0)}{2\cdot 0^2+1}\delta(t) = 0\,\delta(t) = 0$

(b) $\dfrac{j\omega-3}{\omega^2+9}\delta(\omega) = \dfrac{j0-3}{0^2+9}\delta(\omega) = \dfrac{-1}{3}\delta(\omega)$

(c) $\left[e^{-t}\cos(3t-\frac{\pi}{3})\right]\delta(t+\pi) = e^{-(-\pi)}\cos\left(3(-\pi)-\frac{\pi}{3}\right)\delta(t+\pi) = e^{\pi}\cos(\frac{4}{3}\pi)\,\delta(t+\pi) = \frac{-1}{2}e^{\pi}\delta(t+\pi$

(d) $\dfrac{\sin\pi(t+2)}{t^2-4}\delta(t-1) = \dfrac{\sin\pi(1+2)}{1^2-4}\delta(t-1) = \dfrac{0}{-3}\delta(t-1) = 0$

(e) $\dfrac{\cos(\pi t)}{t+2}\delta(2t+3) = \dfrac{\cos(\pi\cdot\frac{-3}{2})}{(\frac{-3}{2})+2}\delta(2t+3) = \dfrac{0}{(\frac{1}{2})}\delta(2t+3) = 0$

(f) $\dfrac{\sin k\omega}{\omega}\delta(\omega) = \dfrac{k\cos(k\omega)}{1}\delta(\omega) = k\cos(k\cdot 0)\,\delta(\omega) = k\,\delta(\omega)$

L'Hôpital's rule

6. $\boxed{2.5\text{-}5}$ $\displaystyle\int_{-\infty}^{\infty}|x(t)|^2 dt = E_x$ . $\displaystyle\int_{-\infty}^{\infty}|y(t)|^2 dt = E_y$

(a)

$$E_{x+y} = \int_{-\infty}^{\infty}|x(t)+y(t)|^2 dt = \int_{-\infty}^{\infty}(x(t)+y(t))(x^*(t)+y^*(t))\,dt$$

$$= \int_{-\infty}^{\infty}|x(t)|^2 dt + \int_{-\infty}^{\infty}|y(t)|^2 dt + \cancelto{0}{\int_{-\infty}^{\infty}x(t)y^*(t)\,dt} + \cancelto{0}{\int_{-\infty}^{\infty}x^*(t)y(t)\,dt}$$

$$= E_x + E_y + 0 + 0 = E_x + E_y$$

$$E_{x-y} = \int_{-\infty}^{\infty}|x(t)-y(t)|^2 dt = \int_{-\infty}^{\infty}(x(t)-y(t))(x^*(t)-y^*(t))\,dt$$

$$= \int_{-\infty}^{\infty}|x(t)|^2 dt + \int_{-\infty}^{\infty}|y(t)|^2 dt - \cancelto{0}{\int_{-\infty}^{\infty}x(t)y^*(t)\,dt} - \cancelto{0}{\int_{-\infty}^{\infty}x^*(t)y(t)\,d}$$

$$= E_x + E_y - 0 - 0 = E_x + E_y$$

$$\therefore\ E_{x+y} = E_{x-y} = E_x + E_y$$

(b)

$$E_{c_1x+c_2y} = \int_{-\infty}^{\infty}|c_1x(t)+c_2y(t)|^2 dt = \int_{-\infty}^{\infty}(c_1x(t)+c_2y(t))(c_1^*x^*(t)+c_2^*y^*(t))\,dt$$

$$= |c_1|^2E_x + |c_2|^2E_y + c_1c_2^*\cancelto{0}{\int_{-\infty}^{\infty}x(t)y^*(t)\,dt} + c_1^*c_2\cancelto{0}{\int_{-\infty}^{\infty}x^*(t)y(t)\,dt}$$

$$= |c_1|^2E_x + |c_2|^2E_y$$

$$E_{c_1x-c_2y} = \int_{-\infty}^{\infty}|c_1x(t)-c_2y(t)|^2 dt = \int_{-\infty}^{\infty}(c_1x(t)-c_2y(t))(c_1^*x^*(t)-c_2^*y^*(t))\,dt$$

$$= |c_1|^2E_x + |c_2|^2E_y - c_1c_2^*\cancelto{0}{\int_{-\infty}^{\infty}x(t)y^*(t)\,dt} - c_1^*c_2\cancelto{0}{\int_{-\infty}^{\infty}x^*(t)y(t)\,dt}$$

$$= |c_1|^2E_x + |c_2|^2E_y$$

(c)

$$E_{xy} = \int_{-\infty}^{\infty}x(t)y^*(t)\,dt$$

$$E_z = E_{x\pm y} = \int_{-\infty}^{\infty}|x(t)\pm y(t)|^2 dt = \int_{-\infty}^{\infty}(x(t)\pm y(t))(x^*(t)\pm y^*(t))\,dt$$

$$= \int_{-\infty}^{\infty}|x(t)|^2 dt + \int_{-\infty}^{\infty}|y(t)|^2 dt \pm \int_{-\infty}^{\infty}x(t)y^*(t)\,dt \pm \int_{-\infty}^{\infty}x^*(t)y(t)\,d$$

$$= E_x + E_y \pm E_{xy} \pm E_{yx}$$

$$= E_x + E_y \pm (E_{xy} + E_{yx})$$

7 $\boxed{2.8\text{-}4}$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} g(t)\,dt = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} g(t)\,dt + \frac{1}{T_0}\int_{\frac{T_0}{2}}^{T_0} g(t)\,dt = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} g(t)\,dt + \frac{1}{T_0}\int_0^{\frac{T_0}{2}} g(t+\tfrac{T_0}{2})\,dt$$

$$= \frac{1}{T_0}\int_0^{\frac{T_0}{2}} g(t) - g(t)\,dt = 0$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} g(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} g(t)\cos(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{\frac{T_0}{2}}^{T_0} g(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} g(t)\cos(n\omega_0 t)\,dt + \frac{2}{T_0}\int_0^{\frac{T_0}{2}} g(t+\tfrac{T_0}{2})\cos(n\omega_0(t+\tfrac{T_0}{2}))\,dt$$

$\frac{n\omega_0 T_0}{2} = \frac{n2\pi f_0 T_0}{2} = n\pi$

$$= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} \left(g(t) + g(t+\tfrac{T_0}{2})(-1)^n\right)\cos(n\omega_0 t)\,dt = \begin{cases} 0 & , n \text{ even} \\ \frac{4}{T_0}\int_0^{\frac{T_0}{2}} g(t)\cos(n\omega_0 t)\,dt & , n \text{ odd} \end{cases}$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} g(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} g(t)\sin(n\omega_0 t)\,dt + \frac{2}{T_0}\int_{\frac{T_0}{2}}^{T_0} g(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} g(t)\sin(n\omega_0 t)\,dt + \frac{2}{T_0}\int_0^{\frac{T_0}{2}} g(t+\tfrac{T_0}{2})\sin(n\omega_0(t+\tfrac{T_0}{2}))\,dt$$

$$= \frac{2}{T_0}\int_0^{\frac{T_0}{2}} \left(g(t) + g(t+\tfrac{T_0}{2})(-1)^n\right)\sin(n\omega_0 t)\,dt = \begin{cases} 0 & , n \text{ even} \\ \frac{4}{T_0}\int_0^{\frac{T_0}{2}} g(t)\sin(n\omega_0 t)\,dt & , n \text{ odd} \end{cases}$$

Figure P 2.8-6 (a) $g(t) = -g(t-\frac{T_0}{2})$ , $T_0 = 8$ , $\omega_0 = 2\pi\frac{1}{T_0} = \frac{\pi}{4}$ , half-wave symmetric.

Use above results $a_0 = 0$

When $n$ is even, $a_n = b_n = 0$

When $n$ is odd, $a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} g(t)\cos(n\omega_0 t)\,dt$

$$= \frac{4}{8}\int_0^{4} g(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{4}{8}\int_0^{2} \frac{t}{2}\cos(n\omega_0 t)\,dt$$

$$= \left(\frac{t}{4n\omega_0}\sin(n\omega_0 t) + \frac{1}{4n^2\omega_0^2}\cos(n\omega_0 t)\right)\Bigg|_0^2$$

$\omega_0 = \frac{\pi}{4}$

$$= \frac{4}{\pi^2 n^2}\left(\frac{n\pi}{2}\sin(\frac{n\pi}{2}) - 1\right) = \begin{cases} \frac{4}{\pi^2 n^2}\left(\frac{n\pi}{2} - 1\right), & n = 1,5,9 \\ \frac{-4}{\pi^2 n^2}\left(\frac{n\pi}{2} + 1\right), & n = 3,7,1 \end{cases}$$

Figure P2.8-6(a) continued

when $n$ is odd, $b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} g(t)\sin(n\omega_0 t)\,dt$

$$= \frac{4}{8}\int_0^4 g(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{4}{8}\int_0^2 \frac{t}{2}\sin(n\omega_0 t)\,dt$$

$$= \left(\frac{-t}{4n\omega_0}\cos(n\omega_0 t) + \frac{1}{4n^2\omega_0^2}\sin(n\omega_0 t)\right)\Bigg|_0^2$$

$$= \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} \frac{4}{n^2\pi^2}, & n = 1, 5, 9, \cdots \\ -\frac{4}{n^2\pi^2}, & n = 3, 7, 11, \cdots \end{cases}$$

$$\boxed{\therefore\ g(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos\left(\frac{n\pi}{4}t\right) + b_n\left(\frac{n\pi}{4}t\right)}$$

7. $\boxed{2.8\text{-}4}$ Continued

Figure P2.8-6(b) $T_0 = 2\pi$, $\omega_0 = 2\pi\frac{1}{T_0} = 1$, $g(t) = -g\left(t - \frac{T_0}{2}\right)$

half-wave symmetric

Use above results: $a_0 = 0$

when $n$ is even, $a_n = b_n = 0$

when $n$ is odd,

$$a_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} g(t)\cos(n\omega_0 t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\cos(nt)\,dt$$

$$= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + \frac{1}{100}}\left(\frac{-1}{10}\cos(nt) + n\sin(nt)\right)\right]\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\pi/10}}{n^2 + \frac{1}{100}}\cdot\frac{-1}{10}(-1)^n - \frac{e^0}{n^2 + \frac{1}{100}}\cdot\frac{-1}{10}\right] = \frac{e^{-\frac{\pi}{10}} + 1}{5\pi\left(n^2 + \frac{1}{100}\right)}$$

$$b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} g(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-t/10}\sin(nt)\,dt$$

$$= \frac{2}{\pi}\left[\frac{e^{-t/10}}{n^2 + \frac{1}{100}}\left(\frac{-1}{10}\sin(nt) - n\cos(nt)\right)\right]\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\pi/10}}{n^2 + \frac{1}{100}}n + \frac{e^0}{n^2 + \frac{1}{100}}n\right] = \frac{2n}{\pi\left(n^2 + \frac{1}{100}\right)}\left(e^{-\pi/10} + 1\right)$$

$$\boxed{\therefore\ g(t) = \sum_{n=1,3,5,\cdots}^{\infty} a_n\cos(nt) + b_n\sin(nt)}$$

8. $\boxed{2.9\text{-}1}$

(a) $D_0 = \frac{1}{T_0}\int_{T_0} g(t)\,dt = 0$ , $T_0 = 4$.

$$\underset{\substack{\uparrow \\ n\neq 0}}{D_n} = \frac{1}{T_0}\int_{T_0} g(t)e^{-j\frac{2\pi n}{T_0}t}dt = \frac{1}{4}\int_{-1}^{1} 1\, e^{-j\frac{n}{2}\pi t}dt + \frac{1}{4}\int_{1}^{3}(-1)e^{-j\frac{n\pi}{2}t}dt$$

$$= \frac{1}{-2n\pi j}\left[e^{-j\frac{n\pi}{2}t}\Big|_{-1}^{1}\right] + \frac{1}{2n\pi j}\left[e^{-j\frac{n\pi}{2}t}\Big|_{1}^{3}\right]$$

$$= \frac{1}{-2n\pi j}(-2j)\sin\left(\frac{\pi}{2}n\right) + \frac{1}{2n\pi j}e^{-jn\pi}(-2j)\sin\left(\frac{\pi}{2}n\right)$$

$$= \frac{1}{n\pi}\sin\left(\frac{\pi}{2}n\right) + \frac{1}{n\pi}\sin\left(\frac{\pi}{2}n\right)(-1)^{n+1} = \begin{cases} 0 & , n \text{ even} \\ \frac{2}{n\pi}\sin\left(\frac{\pi}{2}n\right) & , n \text{ odd}\end{cases}$$

(b) $T_0 = 10\pi$

$$D_0 = \frac{1}{10\pi}\int_{-\pi}^{\pi} 1\,dt = \frac{1}{5}$$

$$\underset{\substack{\uparrow \\ n\neq 0}}{D_n} = \frac{1}{10\pi}\int_{-\pi}^{\pi} 1\cdot e^{-j\frac{2\pi n}{10\pi}t}dt = \frac{1}{10\pi}\int_{-\pi}^{\pi} e^{-j\frac{n}{5}t}dt = \frac{1}{10\pi}\cdot\frac{1}{-j\frac{n}{5}}\left[e^{-j\frac{n}{5}\pi} - e^{j\frac{n}{5}\pi}\right]$$

$$= \frac{1}{10\pi}\cdot\frac{1}{-j\frac{n}{5}}(-2j)\sin\left(\frac{n\pi}{5}\right)$$

$$= \frac{1}{n\pi}\sin\left(\frac{n\pi}{5}\right), \quad n\neq 0$$

(c) $D_0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}dt = \frac{1}{2}$ , $T_0 = 2\pi$

$$\underset{\substack{\uparrow \\ n\neq 0}}{D_n} = \frac{1}{2\pi}\int_0^{2\pi}\frac{t}{2\pi}e^{-j\frac{2\pi}{2\pi}nt}dt = \frac{1}{4\pi^2}\int_0^{2\pi} t e^{-jnt}dt = \frac{-1}{4\pi^2}\left[\frac{1}{jn}te^{-jnt} + \frac{1}{n^2}e^{-jnt}\Big|_0^{2\pi}\right] = \frac{j}{2n\pi}, \quad n\neq 0$$

(d) $D_0 = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{4}{\pi}t\,dt = 0$ , $T_0 = \pi$

$$\underset{\substack{\uparrow \\ n\neq 0}}{D_n} = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4}\frac{4}{\pi}te^{-j\frac{2\pi}{\pi}nt}dt = \frac{4}{\pi^2}\left[\frac{1}{-j2n}\cdot\frac{\pi}{4}e^{-j\frac{\pi}{2}n} - \frac{1}{4n^2}e^{-j\frac{\pi}{2}n} + \frac{1}{j2n}\left(\frac{-\pi}{4}\right)e^{j\frac{\pi}{2}n} + \frac{1}{4n^2}e^{j\frac{\pi}{2}n}\right]$$

$$= \frac{j}{\pi n}\cos\left(\frac{\pi}{2}n\right) - \frac{2j}{\pi^2 n^2}\sin\left(\frac{\pi}{2}n\right), \quad n\neq 0$$

(e) $D_0 = \frac{1}{3}\int_0^1 t\,dt = \frac{1}{6}$ , $T_0 = 3$

$$\underset{\substack{\uparrow \\ n\neq 0}}{D_n} = \frac{1}{3}\int_0^1 te^{-j\frac{2\pi}{3}nt}dt = \frac{1}{3}\left[\frac{1}{-j\frac{2\pi}{3}n}e^{-j\frac{2\pi}{3}n} + \frac{1}{\frac{4\pi^2}{9}n^2}\left[e^{-j\frac{2\pi}{3}n} - 1\right]\right]$$

$$= \frac{3}{4\pi^2 n^2}\left[e^{-j\frac{2\pi}{3}n} - 1\right] + \frac{j}{2\pi n}e^{-j\frac{2\pi}{3}n}, \quad n\neq 0$$

(f) $D_0 = \frac{1}{6}\int_{-2}^{-1}(t+2)dt + \frac{1}{6}\int_{-1}^{1} 1\,dt + \frac{1}{6}\int_1^2(2-t)dt = \frac{1}{2}$ , $T_0 = 6$

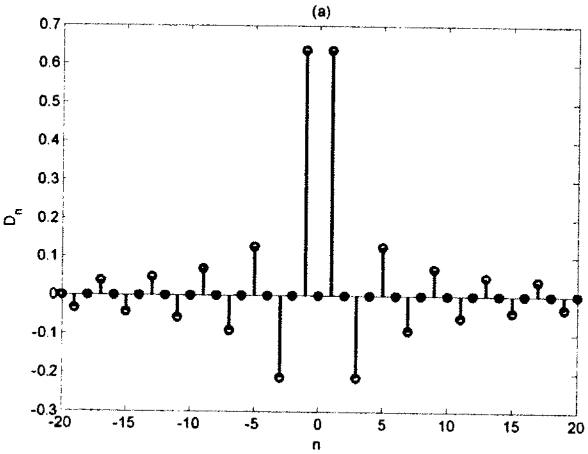
$$\underset{\substack{\uparrow \\ n\neq 0}}{D_n} = \frac{1}{6}\int_{-2}^{-1}(t+2)e^{-j\frac{2\pi}{6}nt}dt + \frac{1}{6}\int_{-1}^{1}1\cdot e^{-j\frac{2\pi}{6}nt}dt + \frac{1}{6}\int_1^2(2-t)e^{-j\frac{2\pi}{6}nt}dt$$

$$= \frac{3}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2\pi n}{3}\right), \quad n\neq 0$$

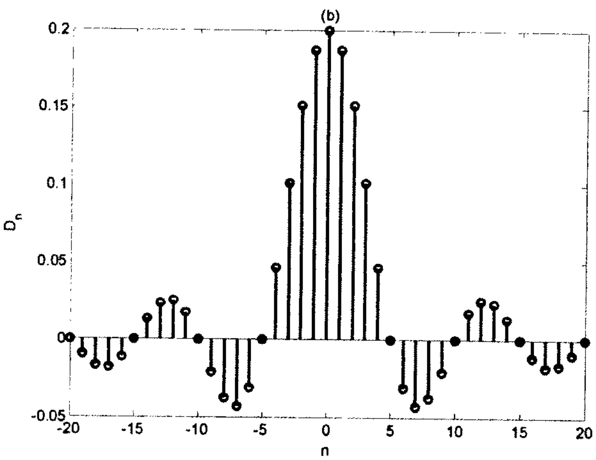
8. $\boxed{2.4\text{-}11}$ continued

Sketch of spectra

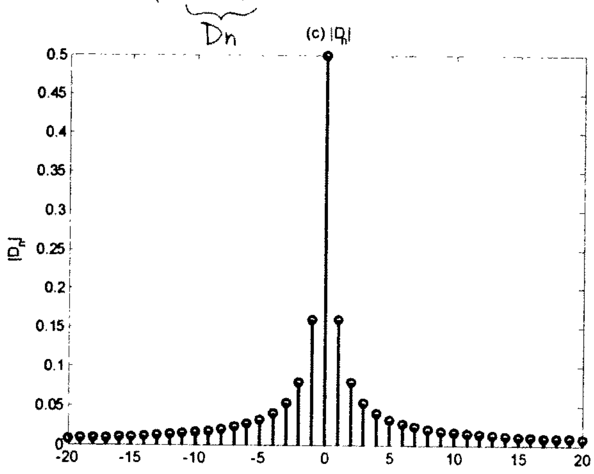
(a) $g(t) = \sum_{n \text{ odd}} \underbrace{\frac{2}{n\pi}\sin\left(\frac{\pi}{2}n\right)}_{D_n} e^{j\frac{n\pi}{2}t}$

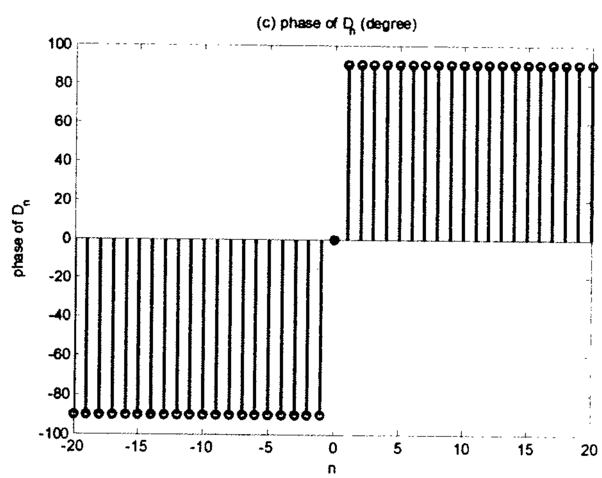



(b) $g(t) = \frac{1}{5} + \sum_{n\neq 0} \underbrace{\frac{1}{n\pi}\sin\left(\frac{n\pi}{5}\right)}_{D_n} e^{j\frac{n}{5}t}$



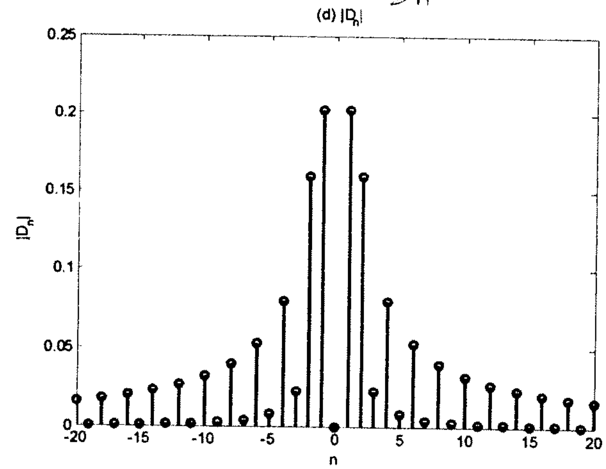

(c) $g(t) = \frac{1}{2} + \sum_{n\neq 0} \underbrace{\frac{j}{2n\pi}}_{D_n} e^{jnt}$

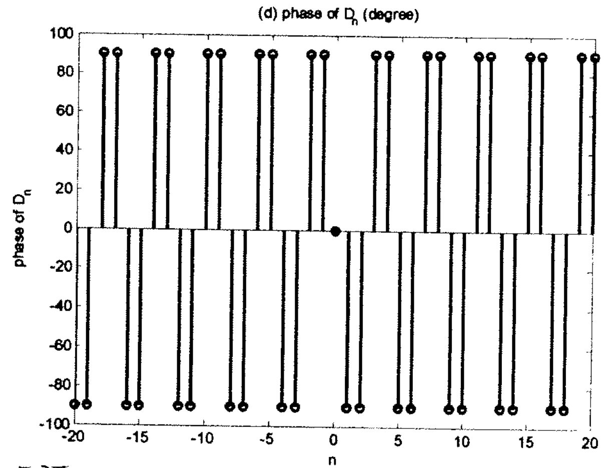


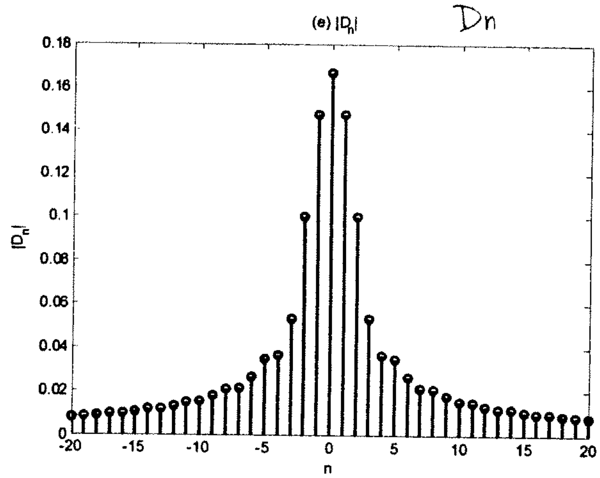
8. $\boxed{2.9\text{-}1}$ continued

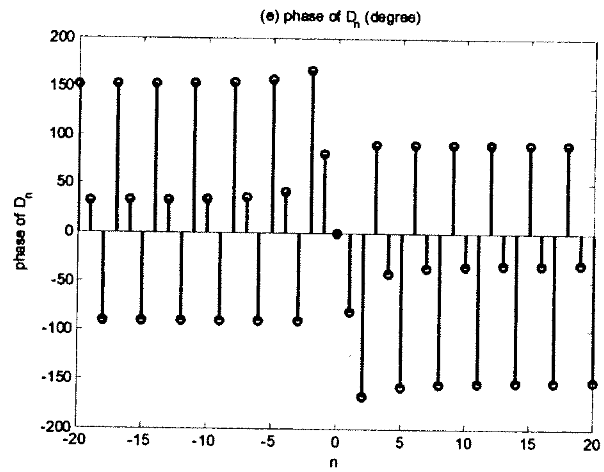
(d) $$g(t) = \sum_{n\neq 0} \underbrace{\left(\frac{j}{\pi n}\cos\left(\frac{\pi}{2}n\right) - \frac{2j}{\pi^2 n^2}\sin\left(\frac{\pi}{2}n\right)\right)}_{D_n} e^{j2nt}$$



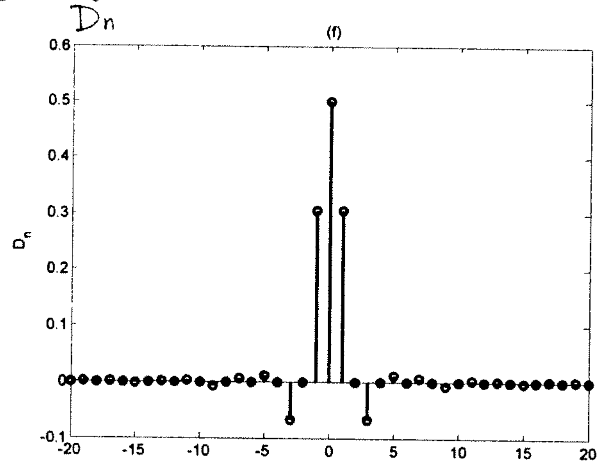




(e) $$g(t) = \frac{1}{6} + \sum_{n\neq 0} \underbrace{\left(\frac{3}{4\pi^2 n^2}\left[e^{-j\frac{2\pi}{3}n} - 1\right] + \frac{j}{2\pi n} e^{-j\frac{2\pi}{3}n}\right)}_{D_n} e^{j\frac{2\pi}{3}nt}$$



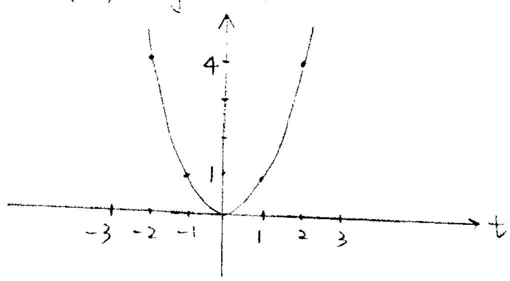




(f) $$g(t) = \frac{1}{2} + \sum_{n\neq 0} \underbrace{\frac{3}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right)}_{D_n} e^{j\frac{\pi}{3}nt}$$



4. $\boxed{2.7\text{-}3}$

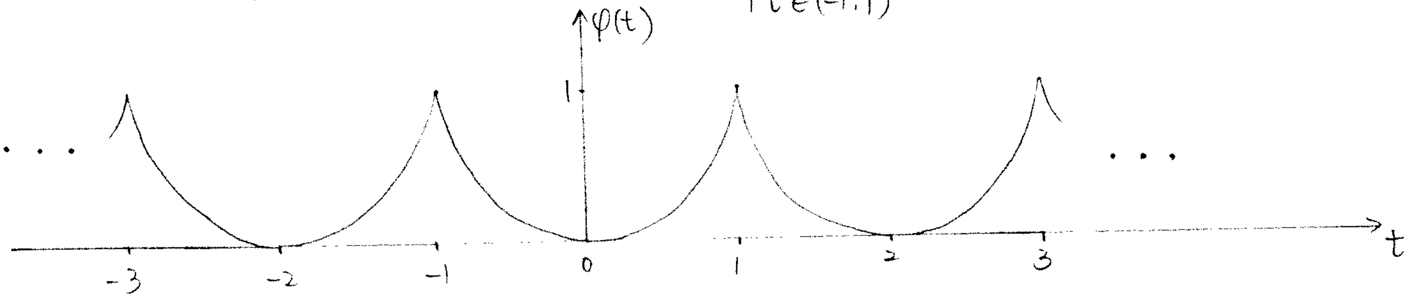
(a) $g(t)=t^2$



$$I=(-1,1)$$
$$T_0=2$$
$$D_0=\frac{1}{2}\int_{-1}^{1}t^2dt=\frac{1}{2}\cdot\frac{1}{3}t^3\Big|_{-1}^{1}=\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$$
$$\underset{n\neq 0}{D_n}=\frac{1}{2}\int_{-1}^{1}t^2e^{-j\frac{2\pi n}{2}t}dt$$
$$=\frac{1}{2}\int_{-1}^{1}t^2e^{-jn\pi t}dt$$
$$=\frac{1}{2}\cdot\frac{1}{-jn\pi}\left[t^2e^{-jn\pi t}+\frac{2te^{-jn\pi t}}{jn\pi}-\frac{2}{n^2\pi^2}e^{-jn\pi t}\right]\Bigg|_{-1}^{1}$$
$$=\frac{1}{-2jn\pi}\left[e^{-jn\pi}\left(1+\frac{2}{jn\pi}-\frac{2}{n^2\pi^2}\right)-e^{jn\pi}\left(1-\frac{2}{jn\pi}-\frac{2}{n^2\pi^2}\right)\right]$$
$$=\frac{1}{-2jn\pi}\left[\cancel{(-2j)\sin(n\pi)}^{\,0}+\frac{2}{jn\pi}2\cos(n\pi)+\cancel{\frac{2}{n^2\pi^2}(2j)\sin(n\pi)}^{\,0}\right]$$
$$=\frac{2\cos(n\pi)}{n^2\pi^2},\quad n\neq 0$$

$$\therefore\ g(t)\Big|_{t\in(-1,1)}=\frac{1}{3}+\sum_{n\neq 0}^{\infty}D_ne^{j\pi nt}=\frac{1}{3}+\sum_{n\neq 0}^{\infty}\frac{2\cos(n\pi)}{n^2\pi^2}\left[\cos(n\pi t)+j\sin(n\pi t)\right]$$
$$=\frac{1}{3}+2\sum_{n=1}^{\infty}\frac{2\cos(n\pi)\cos(n\pi t)}{n^2\pi^2}=\frac{1}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi}{n^2\pi^2}$$

$\varphi(t)$ is periodic extension of $g(t)\Big|_{t\in(-1,1)}$:




(b) verify Parseval's Theorem [(2.102a)]:

$$P_g=\frac{1}{2}\int_{-1}^{1}(t^2)^2dt=\frac{1}{2}\int_{-1}^{1}t^4dt=\frac{1}{2}\cdot\frac{1}{5}t^5\Big|_{-1}^{1}=\frac{1}{5}$$
$$\sum_{n=-\infty}^{\infty}|D_n|^2=|D_0|^2+\sum_{n\neq 0}|D_n|^2=\left(\frac{1}{3}\right)^2+\sum_{n\neq 0}\frac{2^2\cos^2(n\pi)}{n^4\pi^4}=\frac{1}{9}+2\sum_{n=1}^{\infty}\frac{4(-1)^{2n}}{n^4\pi^4}$$
$$=\frac{1}{9}+8\times\frac{1}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{9}+8\times\frac{1}{\pi^4}\times\frac{\pi^4}{90}=\frac{1}{5}$$

We have $P_g=\sum_{n=-\infty}^{\infty}|D_n|^2=\frac{1}{5}$, (2.102a) is verified.

**10.**

A signal $g : \mathbb{R} \to \mathbb{R}$ is of the form

$$g(t) = \cos(2\pi t) + \sin(6\pi t), \quad t \in \mathbb{R}.$$

Can you find its Fourier series expansion without doing any computations? Explain. Generalize to the signal

$$g(t) = \sum_{k=0}^{K} a_k \cos(2\pi k f t) + \sum_{k=1}^{K} \overset{b_k}{\cancel{a_k}} \sin(2\pi k f t), \quad t \in \mathbb{R}$$

with $f > 0$, positive integer $K$ and scalars $a_0, \ldots, a_K, b_1, \ldots, b_K$.

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Yes. Let $g(t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(\omega_0 n t) + \sum_{n=1}^{\infty} \beta_n \sin(\omega_0 n t)$ be the Fourier series expansion of $g(t)$.

(1) For $g(t) = \cos(2\pi t) + \sin(6\pi t)$, Fourier series exponsion is $\alpha_1 \cos(2\pi t) + \beta_3 \sin(6\pi t)$, with $\alpha_1 = \beta_3 = 1$ and $\begin{cases} \alpha_n = 0, & \forall n \neq 1 \\ \beta_n = 0, & \forall n \neq 3 \end{cases}$

Explain. We can write $g(t) = \cos(\omega_0 t) + \sin(3\omega_0 t)$, where $\omega_0 = 2\pi\ \left(\frac{rad}{s}\right)$ is the fundamental frequency of $g(t)$; by comparison we know $\alpha_1 = 1$, $\beta_3 = 1$, and other $\alpha_n, \beta_n$ are zero.

(2) For $g(t) = \sum_{k=0}^{K} a_k \cos(2\pi k f t) + \sum_{k=1}^{K} b_k \sin(2\pi k f t)$, Fourier series expansion is $\sum_{k=0}^{K} \alpha_k \cos(2\pi k f t) + \sum_{k=1}^{K} \beta_k \sin(2\pi k f t)$, with

$$\alpha_k = a_k \text{ and } \beta_k = b_k \quad \forall k$$

Explain. We can write $g(t) = \sum_{k=0}^{K} a_k \cos(k\omega_0 t) + \sum_{k=1}^{K} b_k \sin(k\omega_0 t)$, where $\omega_0 = 2\pi f\ \left(\frac{rad}{s}\right)$ is the fundamental frequency of $g(t)$; by comparison we know $\alpha_k = a_k$, $\beta_k = b_k$ $\forall k$