HW 2 Solutions

1. $\boxed{3.1\text{-}1}$ $G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt=\int_{-\infty}^{\infty} g(t)[\cos(2\pi ft)-j\sin(2\pi ft)]\,dt=\int_{-\infty}^{\infty} g(t)\cos(2\pi ft)dt$
$-j\int_{-\infty}^{\infty} g(t)\sin(2\pi ft)dt$

If $g(t)$ is even function of t ($g(t)=g(-t)$), then
$g(t)\cos(2\pi ft)$ is an even function of t, and
$g(t)\sin(2\pi ft)$ is an odd function of t. Hence $G(f)=2\int_0^{\infty} g(t)\cos(2\pi ft)\,dt$

If $g(t)$ is an odd function of t ($g(t)=-g(-t)$), then
$g(t)\cos(2\pi ft)$ is an odd function of t, and
$g(t)\sin(2\pi ft)$ is an even function of t. Hence $G(f)=-2j\int_0^{\infty} g(t)\sin(2\pi ft)dt$

Proves of properties:

① $g(t)=g^*(t)$, $g(t)=g(-t)\Rightarrow\begin{cases}G(-f)=2\int_0^{\infty} g(t)\cos(2\pi(-f)t)\,dt=G(f)\\ G^*(f)=2\int_0^{\infty} g^*(t)\cos^*(2\pi ft)\,dt=G(f)\end{cases}$ real & even

② $g(t)=g^*(t)$, $g(t)=-g(-t)\Rightarrow\begin{cases}G(-f)=-2j\int_0^{\infty} g(t)\sin(2\pi(-f)t)dt=-G(f)\\ G^*(f)=(-2j)^*\int_0^{\infty} g^*(t)\sin^*(2\pi ft)\,dt=-G(f)\end{cases}$ imaginary & odd

③ $g(t)=-g^*(t)$, $g(t)=g(-t)\Rightarrow\begin{cases}G(-f)=2\int_0^{\infty} g(t)\cos(2\pi(-f)t)dt=G(f)\\ G^*(f)=2\int_0^{\infty} g^*(t)\cos^*(2\pi ft)\,dt=-G(f)\end{cases}$ imaginary & even

④ $g(t)=g(-t)\Rightarrow G(-f)=\int_{-\infty}^{\infty} g(t)e^{j2\pi ft}dt=\int_{-\infty}^{\infty} g(-t)e^{-j2\pi ft}dt=G(f)$ complex & even

⑤ $g(t)=-g(-t)\Rightarrow G(-f)=\int_{-\infty}^{\infty} g(t)e^{j2\pi ft}dt=\int_{\infty}^{\infty} g(-t)e^{-j2\pi ft}dt=-G(f)$ complex & odd

2. $\boxed{3.1\text{-}2}$

(a) $G(f)=\int_{-\infty}^{3}\exp(2(t-3))\,e^{-j2\pi ft}dt+\int_{3}^{\infty}\exp(-2(t-3))\,e^{-j2\pi ft}dt$

$=e^{-6}\int_{-\infty}^{3} e^{(2-j2\pi f)t}dt+e^{6}\int_{3}^{\infty} e^{-(2+j2\pi f)t}dt$

$=e^{-6}\frac{1}{2-j2\pi f}e^{(2-j2\pi f)\cdot 3}+e^{6}\frac{-1}{2+j2\pi f}\left[0-e^{-(2+j2\pi f)\cdot 3}\right]=\frac{e^{-j6\pi f}}{1+\pi^2f^2}$

(b) $g(t)=\int_{-\infty}^{\infty} G(f)\,e^{j2\pi ft}df$

$=\int_{-\infty}^{\infty}\delta(4\pi f)e^{j2\pi ft}df-\int_{-\infty}^{\infty}\delta(f-2)e^{j2\pi ft}df$

$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\delta(v)e^{j\frac{v}{2}t}dv-\int_{-\infty}^{\infty}\delta(f-2)e^{j2\pi(2)t}df$

$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\delta(v)\cdot 1\,dv-e^{j4\pi t}\int_{-\infty}^{\infty}\delta(f-2)\,df=\frac{1}{4\pi}-e^{j4\pi t}$

3. $\boxed{3.1\text{-}4}$

(a) $$G(f)=\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt = \int_0^1 4e^{-j2\pi ft}dt+\int_1^2 2e^{-j2\pi ft}dt$$
$$= \frac{4}{-j2\pi f}\left[e^{-j2\pi f}-1\right]+\frac{2}{-j2\pi f}\left[e^{-j2\pi f(2)}-e^{-j2\pi f}\right]$$
$$= \frac{1}{j2\pi f}\left[4-2e^{-j2\pi f}-2e^{-j4\pi f}\right]$$

(b) $$G(f)=2\int_0^{\tau}\frac{t}{\tau}\cos(2\pi ft)dt$$
$$=\frac{2}{\tau}\left[t\sin(2\pi ft)\cdot\frac{1}{2\pi f}+\cos(2\pi ft)\cdot\frac{1}{4\pi^2f^2}\right]\Bigg|_0^{\tau}$$
$$=\frac{2}{\tau}\left[\tau\sin(2\pi f\tau)\cdot\frac{1}{2\pi f}+\cos(2\pi f\tau)\frac{1}{4\pi^2f^2}-\frac{1}{4\pi^2f^2}\right]$$
$$=\frac{1}{\tau(2\pi^2f^2)}\left[\cos(2\pi f\tau)+\tau(2\pi f)\sin(2\pi f\tau)-1\right]$$

4. $\boxed{3.1\text{-}5}$

(a) $$g(t)=\int_{-\infty}^{\infty}G(f)e^{j2\pi ft}df=2\int_0^B 3f^2\cos(2\pi ft)df$$
$$=\frac{6}{2\pi t}\left[\sin(2\pi ft)f^2+\frac{2f}{2\pi t}\cos(2\pi ft)-\frac{2}{4\pi^2t^2}\sin(2\pi ft)\right]\Bigg|_0^B$$
$$=\frac{3}{\pi t}\left[\sin(2\pi Bt)B^2+\frac{B}{\pi t}\cos(2\pi Bt)-\frac{1}{2\pi^2t^2}\sin(2\pi Bt)\right]$$

(b) $$G(f)=\text{rect}\left(\frac{f}{2}\right)+\text{rect}\left(\frac{f}{4}\right)$$
$$\Rightarrow g(t)=\int_{-1}^{1}e^{j2\pi ft}df+\int_{-2}^{2}e^{j2\pi ft}df$$
$$=2\int_0^1\cos(2\pi ft)df+2\int_0^2\cos(2\pi ft)df$$
$$=\frac{1}{\pi t}\left[\sin(2\pi t)+\sin(4\pi t)\right]$$

5. $\boxed{3.1\text{-}6}$

(a) $$g(t)=\int_{-\infty}^{\infty}G(f)e^{j2\pi ft}df=2\int_0^{\frac{1}{2}}\cos(\pi f)\cos(2\pi ft)df$$
$$=\int_0^{\frac{1}{2}}\cos((\pi+2\pi t)f)df+\int_0^{\frac{1}{2}}\cos((\pi-2\pi t)f)df$$
$$=\frac{1}{\pi+2\pi t}\sin((\pi+2\pi t)f)\Big|_0^{\frac{1}{2}}+\frac{1}{\pi-2\pi t}\sin((\pi-2\pi t)f)\Big|_0^{\frac{1}{2}}$$
$$=\frac{1}{\pi+2\pi t}\sin\left(\frac{\pi}{2}+\pi t\right)+\frac{1}{\pi-2\pi t}\sin\left(\frac{\pi}{2}-\pi t\right)=\frac{2\cos(\pi t)}{\pi(1-4t^2)}$$

5. $\boxed{3.1-6}$ Continued

(b) $$g(t)=\int_{-\infty}^{\infty}G(f)e^{j2\pi ft}df = 2\int_0^B \frac{f}{B}\cos(2\pi ft)df$$

$$= \frac{2}{B}\left[f\sin(2\pi tf)\cdot\frac{1}{2\pi t} + \cos(2\pi tf)\cdot\frac{1}{4\pi^2t^2}\right]\Big|_0^B$$

$$= \frac{2}{B}\left[B\sin(2\pi Bt)\cdot\frac{1}{2\pi t} + \cos(2\pi Bt)\cdot\frac{1}{4\pi^2t^2} - \frac{1}{4\pi^2t^2}\right]$$

$$= \frac{1}{B(2\pi^2t^2)}\left[\cos(2\pi Bt) + 2\pi Bt\sin(2\pi Bt) - 1\right]$$

6. $\boxed{3.1-7}$ (a) $$G(f)=\begin{cases} e^{-j2\pi ft_0} & , |f|\le B \\ 0 & , \text{else}\end{cases}$$

$$g(t)=\int_{-B}^{+B} e^{-j2\pi ft_0}\cdot e^{j2\pi ft}df$$

$$= \int_{-B}^{B} e^{j2\pi(t-t_0)f}df$$

$$= \frac{1}{j2\pi(t-t_0)}\left[e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}\right]$$

$$= \frac{\sin(2\pi B(t-t_0))}{\pi(t-t_0)}$$

(b) $$G(f)=\begin{cases} -j & , 0<f\le B \\ j & , -B\le f<0\end{cases}$$

$$g(t) = \int_{-B}^{0} j\, e^{j2\pi ft}df + \int_0^B (-j)e^{j2\pi ft}df$$

$$= 2j\int_0^B (-j)\sin(2\pi ft)df$$

$$= 2\int_0^B \sin(2\pi ft)df = \frac{1}{\pi t}\left[1-\cos(2\pi Bt)\right]$$

$\therefore$ $g(t)$ in (a) $\neq$ $g(t)$ in (b)

7. $\boxed{3.3-2}$

(a) $$g(t) \underset{F^{-1}}{\overset{F}{\rightleftharpoons}} G(f)=\frac{1}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right)$$

(b) $$g_1(t)=g(-t) \Rightarrow G_1(f)=G(-f)=\frac{1}{(2\pi f)^2}\left(e^{-j2\pi f}+j2\pi f e^{-j2\pi f}-1\right)$$

(c) $$g_2(t)=g(t-1)+g_1(t-1)=g(t-1)+g(-(t-1))$$

$$\Rightarrow G_2(f)=G(f)e^{-j2\pi f}+G(-f)e^{-j2\pi f}$$

$$= e^{-j2\pi f}\left[\frac{1}{2\pi^2f^2}\cos(2\pi f) + \frac{1}{\pi f}\sin(2\pi f) - \frac{1}{2\pi^2f^2}\right]$$

7. $\boxed{3.3\text{-}2}$ (d) $g_3(t) = g(t-1) + g_1(t+1) = g(t-1) + g(-(t+1))$

$$\Rightarrow G_3(f) = G(f)e^{-j2\pi f} + G(-f)e^{j2\pi f}$$

$$= \frac{1}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right)e^{-j2\pi f} + \frac{1}{(2\pi f)^2}\left(e^{-j2\pi f} + 2j\pi f e^{-j2\pi f} - 1\right)e^{j2\pi f}$$

$$= \frac{1}{(2\pi f)^2}\left[1 - \cancel{j2\pi f} - e^{-j2\pi f} + 1 + \cancel{2j\pi f} - e^{j2\pi f}\right]$$

$$= \frac{1}{2\pi^2 f^2}\left[1 - \cos(2\pi f)\right] = \frac{1}{\pi^2 f^2}\sin^2(\pi f)$$

(e) $g_4(t) = g(t-\frac{1}{2}) + g_1(t+\frac{1}{2})$

$$\Rightarrow G_4(f) = \frac{1}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right)e^{-j\pi f} + \frac{1}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right)e^{j\pi f}$$

$$= \frac{1}{(2\pi f)^2}\left(\cancel{e^{j\pi f}} - j2\pi f e^{j\pi f} - \cancel{e^{-j\pi f}} + \cancel{e^{-j\pi f}} + j2\pi f e^{-j\pi f} - \cancel{e^{j\pi f}}\right)$$

$$= \frac{1}{(2\pi f)^2}(-j2\pi f)(2j\sin(\pi f)) = \frac{\sin(\pi f)}{\pi f}$$

(f) $g_5(t)$ is obtained by (1) (2) (3)

(1) $g(t) \to g(\frac{t}{2}) \underset{F^{-1}}{\overset{F}{\rightleftharpoons}} 2G(2f)$

(2) $g(\frac{t}{2}) \to g(\frac{t-2}{2}) \underset{F^{-1}}{\overset{F}{\rightleftharpoons}} 2G(2f)e^{-j4\pi f}$

(3) $g(\frac{t-2}{2}) \to 1.5\,g(\frac{t-2}{2}) = g_5(t) \underset{F^{-1}}{\overset{F}{\rightleftharpoons}} G_5(f) = 1.5 \times 2G(2f)e^{-j4\pi f}$

$$\Rightarrow G_5(f) = 3G(2f)e^{-j4\pi f} = \frac{3}{(4\pi f)^2}\left[1 - j4\pi f - e^{-j4\pi f}\right]$$

8. $\boxed{3.3\text{-}3}$

(a) $\text{rect}(t) \rightleftharpoons \text{sinc}(f)$, where $\text{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$

$\text{rect}(\frac{t}{T}) \rightleftharpoons T\,\text{sinc}(fT)$

(i) $g(t) = \text{rect}(\frac{t+T/2}{T}) - \text{rect}(\frac{t-T/2}{T}) \Rightarrow G(f) = T\,\text{sinc}(fT)e^{j\pi fT} - T\,\text{sinc}(fT)e^{-j\pi fT}$

$$= 2jT\,\text{sinc}(fT)\sin(\pi fT)$$

(ii) $$G(f) = \int_{-T}^{0} e^{-j2\pi ft}dt + \int_0^T (-1)e^{-j2\pi ft}dt$$

$$= -2j\int_0^T (-1)\sin(2\pi ft)\,dt$$

$$= \frac{-2j}{2\pi f}\cos(2\pi ft)\Big|_0^T$$

$$= \frac{j}{\pi f}\left[1 - \cos(2\pi fT)\right] = \frac{2j}{\pi f}\sin(\pi fT)\sin(\pi fT) = 2jT\,\text{sinc}(fT)\sin(\pi fT)$$

8. $\boxed{3.3\text{-}3}$ continued

(b) (i) $\sin t\, u(t) \rightleftharpoons \frac{1}{4j}\left(\delta(f-\frac{1}{2\pi})-\delta(f+\frac{1}{2\pi})\right)+\frac{1}{1-(2\pi f)^2}$

$\sin(t-\pi)u(t-\pi) \rightleftharpoons \left[\frac{1}{4j}\left(\delta(f-\frac{1}{2\pi})-\delta(f+\frac{1}{2\pi})\right)+\frac{1}{1-(2\pi f)^2}\right]e^{-j2\pi f\pi}$

$g(t)=\sin t\, u(t)+\sin(t-\pi)u(t-\pi) \rightleftharpoons G(f)=\left[\frac{1}{4j}\left(\delta(f-\frac{1}{2\pi})-\delta(f+\frac{1}{2\pi})\right)+\frac{1}{1-4\pi^2f^2}\right]\left[1+e^{-j2\pi^2 f}\right]$

$\overset{(*)}{=}\frac{1}{1-(2\pi f)^2}\left[1+e^{-j2\pi^2 f}\right]$

(ii) $G(f)=\int_0^{\pi}\sin(t)e^{-j2\pi ft}dt$

$=\frac{1}{2j}\int_0^{\pi}e^{j(1-2\pi f)t}dt-\frac{1}{2j}\int_0^{\pi}e^{-j(1+2\pi f)t}dt$

$=\frac{1}{2j}\cdot\frac{1}{j(1-2\pi f)}\left[e^{j(1-2\pi f)\pi}-1\right]+\frac{1}{2j}\cdot\frac{1}{j(1+2\pi f)}\left[e^{-j(1+2\pi f)\pi}-1\right]$

$=\frac{1}{1-(2\pi f)^2}\left[1+e^{-j2\pi^2 f}\right]$

(c) (i) $\cos t\, u(t) \rightleftharpoons \frac{1}{4}\left(\delta(f-\frac{1}{2\pi})+\delta(f+\frac{1}{2\pi})\right)+\frac{j2\pi f}{1-4\pi^2f^2}$

$\sin(t-\frac{\pi}{2})u(t-\frac{\pi}{2}) \rightleftharpoons \left[\frac{1}{4j}\left(\delta(f-\frac{1}{2\pi})-\delta(f+\frac{1}{2\pi})\right)+\frac{1}{1-(2\pi f)^2}\right]e^{-j\pi^2 f}$

$g(t)=\cos t\, u(t)+\sin(t-\frac{\pi}{2})u(t-\frac{\pi}{2})$

$\Rightarrow G(f)=\frac{1}{4}\left(\delta(f-\frac{1}{2\pi})+\delta(f+\frac{1}{2\pi})\right)+\frac{j2\pi f}{1-4\pi^2f^2}+\left[\frac{\delta(f-\frac{1}{2\pi})-\delta(f+\frac{1}{2\pi})}{4j}+\frac{1}{1-4\pi^2f^2}\right]e^{-j\pi^2 f}$

$\overset{(**)}{=}\frac{1}{1-4\pi^2f^2}\left[j2\pi f+e^{-j\pi^2 f}\right]$

(ii) $G(f)=\int_0^{\pi/2}\cos(t)e^{-j2\pi ft}dt$

$=\frac{1}{2}\int_0^{\pi/2}e^{j(1-2\pi f)t}dt+\frac{1}{2}\int_0^{\pi/2}e^{-j(1+2\pi f)t}dt$

$=\frac{1}{2}\frac{1}{j(1-2\pi f)}\left[e^{j(1-2\pi f)\pi/2}-1\right]-\frac{1}{2}\cdot\frac{1}{j(1+2\pi f)}\left[e^{-j(1+2\pi f)\pi/2}-1\right]$

$=\frac{1}{1-(2\pi f)^2}\left[j2\pi f+e^{-j\pi^2 f}\right]$

Note: $(*)$ $\delta(f\pm\frac{1}{2\pi})(1+e^{-j2\pi^2 f})=0$

$(**)$ $\delta(f\pm\frac{1}{2\pi})e^{-j\pi^2 f}=\pm j\,\delta(f\pm\frac{1}{2\pi})$

8. $\boxed{3.3\text{-}3}$ continued

(d) (i) $e^{-at}u(t) \overset{F}{\underset{F^{-1}}{\rightleftharpoons}} \dfrac{1}{a+j2\pi f}$, $e^{-aT}e^{-a(t-T)}u(t-T) \overset{F}{\underset{F^{-1}}{\rightleftharpoons}} \dfrac{e^{-aT}}{a+j2\pi f}e^{-j2\pi fT}$

$$g(t) = e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T) \Rightarrow G(f) = \frac{1}{a+j2\pi f} - \frac{e^{-aT}}{a+j2\pi f}e^{-j2\pi fT}$$

$$= \frac{1}{a+j2\pi f}\left[1-e^{-(a+2\pi fj)T}\right]$$

(ii)
$$G(f) = \int_0^T e^{-at}e^{-j2\pi ft}\,dt$$

$$= \int_0^T e^{-(a+j2\pi f)t}\,dt$$

$$= \frac{-1}{a+j2\pi f}\left[e^{-(a+j2\pi f)T}-1\right] = \frac{1}{a+j2\pi f}\left[1-e^{-(a+j2\pi f)T}\right]$$

9. $\boxed{3.4\text{-}1}$ Let $x_1(t) \rightarrow \boxed{} \rightarrow y_1(t)$, $x_2(t) \rightarrow \boxed{} \rightarrow y_2(t)$. Let $r, s \in \mathbb{R}$, $r \neq 0$, $s \neq 0$

(i.e. $y_i(t)$ is the output when x_i is the input , $i = 1, 2$)

(a) $y_1(t)+y_2(t) = a(x_1(t)+x_2(t))+2b \neq a(x_1(t)+x_2(t))+b$ in general. $\Rightarrow$ <u>NOT linear</u>

(b) $y_1(t)+y_2(t) = A[\cos x_1(t)+\cos x_2(t)] \neq A\cos(x_1(t)+x_2(t))$ $\Rightarrow$ <u>NOT linear</u>

(c)
$$ry_1(t)+sy_2(t) = rC_1x_1(t-t_1)+rC_2x_1(t-t_2)+sC_1x_2(t-t_1)+sC_2x_2(t-t_2)$$
$$= C_1(rx_1(t-t_1)+sx_2(t-t_1))+C_2(rx_1(t-t_2)+sx_2(t-t_2)) \Rightarrow \underline{\text{linear}}$$

(d) $ry_1(t)+sy_2(t) = re^{jw_0t}x_1(t)+se^{jw_0t}x_2(t) = e^{jw_0t}(rx_1(t)+sx_2(t))$ $\Rightarrow$ <u>linear</u>

(e) $ry_1(t)+sy_2(t) = rA\cos(w_0t)x_1(t)+sA\cos(w_0t)x_2(t)$ $\therefore$ linear
$$= A\cos(w_0t)(rx_1(t)+sx_2(t))$$

(f) $ry_1(t)+sy_2(t) = rA\sin(w_0t)x_1(t)+sA\sin(w_0t)x_2(t) = A\sin(w_0t)(rx_1(t)+sx_2(t)) \Rightarrow$ <u>linea</u>

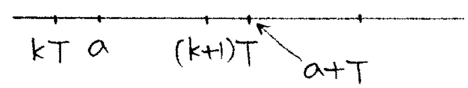
(g) $ry_1(t)+sy_2(t) = rA\cos(w_0t)x_1(t-t_0)-rB\sin(w_0t)x_1(t)$
$$+ sA\cos(w_0t)x_2(t-t_0)-sB\sin(w_0t)x_2(t)$$
$$= A\cos(w_0t)[rx_1(t-t_0)+sx_2(t-t_0)]-B\sin(w_0t)[rx_1(t)+sx_2(t)] \Rightarrow \underline{\text{linea}}$$

(h) $ry_1(t)+sy_2(t) = r\int_{-\infty}^{\infty}Ap(t-\tau)x_1(\tau)d\tau + s\int_{-\infty}^{\infty}Ap(t-\tau)x_2(\tau)d\tau$
$$= \int_{-\infty}^{\infty}Ap(t-\tau)(rx_1(\tau)+sx_2(\tau))\,d\tau \Rightarrow \underline{\text{linear}}$$

(i) $\mathrm{Re}[rx_1(t)+sx_2(t)] = \mathrm{Re}[rx_{1r}(t)+sx_{2r}(t)+j(rx_{1i}(t)+sx_{2i}(t))]$
$$= rx_{1r}(t)+sx_{2r}(t) = r\,\mathrm{Re}[x_1(t)]+s\,\mathrm{Re}[x_2(t)] = ry_1(t)+sy_2(t) \Rightarrow \underline{\text{linear}}$$

(j) $\mathrm{Im}[rx_1(t)+sx_2(t)] = \mathrm{Im}[rx_{1r}(t)+sx_{2r}(t)+j(rx_{1i}(t)+sx_{2i}(t))]$
$$= rx_{1i}(t)+sx_{2i}(t) = r\,\mathrm{Im}[x_1(t)]+s\,\mathrm{Im}[x_2(t)] = ry_1(t)+sy_2(t) \Rightarrow \text{line}$$

10. Given $a \in \mathbb{R}$, we can find an integer k such that $kT \leq a < (k+1)T$



We have $$C_n = \frac{1}{T}\int_a^{a+T} g(t)e^{-j2\pi\frac{n}{T}t}\,dt$$

$$= \frac{1}{T}\int_a^{(k+1)T} g(t)e^{-j2\pi\frac{n}{T}t}\,dt + \frac{1}{T}\int_{(k+1)T}^{a+T} g(t)e^{-j2\pi\frac{n}{T}t}\,dt$$

$\tau = t - T$

$$= \frac{1}{T}\int_a^{(k+1)T} g(t)e^{-j2\pi\frac{n}{T}t}\,dt + \frac{1}{T}\int_{kT}^{a} g(\tau+T)e^{-j2\pi\frac{n}{T}(\tau+T)}\,d\tau$$

$g(\tau+T) = g(\tau)$

$$= \frac{1}{T}\int_a^{(k+1)T} g(t)e^{-j2\pi\frac{n}{T}t}\,dt + \frac{1}{T}\int_{kT}^{a} g(\tau)e^{-j2\pi\frac{n}{T}\tau}\,d\tau$$

$$= \frac{1}{T}\int_{kT}^{(k+1)T} g(t)e^{-j2\pi\frac{n}{T}t}\,dt$$

$\tau = t - kT$

$$= \frac{1}{T}\int_0^{T} g(\tau+kT)e^{-j2\pi\frac{n}{T}(\tau+kT)}\,d\tau$$

$g(\tau+kT) = g(\tau)$
($\because$ $g(\tau)$ is periodic with period T)

$$= \frac{1}{T}\int_0^{T} g(t)e^{-j2\pi\frac{n}{T}t}\,dt, \quad n = 0, \pm1, \pm2, \cdots$$

——— regardless of the value of a #