COMMUNICATION SYSTEMS

HOMEWORK # 2 – An Answer to Ex. 10

Consider a periodic signal $g : \mathbb{R} \to \mathbb{C}$ with period $T > 0$, i.e.,

$$g(t+T) = g(t), \quad t \in \mathbb{R}.$$

Assume that

$$\int_0^T |g(t)| dt < \infty.$$

Show that the Fourier coefficients of g can be computed as

$$c_n = \frac{1}{T} \int_a^{a+T} g(t) e^{-j2\pi \frac{n}{T} t} dt, \quad n = 0, \pm 1, \pm 2, \ldots$$

regardless of the value of a in $\mathbb{R}$.

For any scalar a in $\mathbb{R}$, it is always possible to write $a = \ell T + \alpha$ for some $\ell = 0, \pm 1, \ldots$ and $0 \leq \alpha < T$ – Note that ℓ and α are uniquely determined by a once $T > 0$ is given. Now, for any mapping $h : \mathbb{R} \to \mathbb{C}$ which is periodic with period T, whenever the integrability condition

$$\int_0^T |h(t)| dt < \infty$$

holds, we have

$$\int_a^{a+T} h(t) dt = \int_0^T h(t) dt. \tag{1.1}$$

Indeed, we have

$$\begin{aligned}
\int_a^{a+T} h(t)dt &= \int_0^T h(a+s)ds \quad \text{[Change of variable: } t = a+s\text{]} \\
&= \int_0^T h(\ell T + \alpha + s)ds \quad \text{[Use the fact that } a = \ell T + \alpha\text{]} \\
&= \int_0^T h(\alpha + s)ds \quad \text{[Use the fact that } h \text{ is periodic of period } T\text{]} \\
&= \int_0^\alpha h(\alpha + s)ds + \int_\alpha^T h(\alpha+s)ds \\
&= \int_0^\alpha h(\alpha + s)ds + \int_0^{T-\alpha} h(x)dx \quad \text{[Change of variable } x = \alpha + s\text{]} \\
&= \int_{T-\alpha}^T h(y)dy + \int_0^{T-\alpha} h(x)dx \quad \text{[Change of variable } y = T - \alpha + s\text{]} \\
&= \int_0^T h(x)dx \qquad (1.2)
\end{aligned}$$

and (1.1) is established.

The exercise is solved by applying (1.1) to the functions

$$h_n(t) = g(t)e^{-j2\pi \frac{n}{T}t}, \qquad \begin{array}{c} t \in \mathbb{R} \\ n = 0, \pm 1, 2, \ldots \end{array}$$

where $g : \mathbb{R} \to \mathbb{C}$ is periodic with period T and satisfies the integrability condition

$$\int_0^T |g(t)|dt < \infty.$$

Indeed, for each $n = 0, \pm 1, \pm 2, \ldots$, the function $h_n : \mathbb{R} \to \mathbb{C}$ is periodic of period T since

$$\begin{aligned}
h_n(t+T) &= g(t+T)e^{-j2\pi \frac{n}{T}(t+T)} \\
&= g(t)e^{-j2\pi \frac{n}{T}(t+T)} \quad \text{[Recall that } g \text{ is periodic with period } T\text{]} \\
&= g(t)e^{-j2\pi n}e^{-j2\pi \frac{n}{T}t} \\
&= g(t)e^{-j2\pi \frac{n}{T}t} \\
&= h_n(t), \quad n = 0, \pm 1, \pm 2, \ldots \qquad (1.3)
\end{aligned}$$

Also

$$\int_0^T |h_n(t)|dt = \int_0^T |g(t)|dt < \infty$$

and the integrability condition is satisfied.