1. $\boxed{3.4\text{-}3}$ $H(f)=\dfrac{1}{1+j2\pi f}$ . $h(t)=e^{-t}u(t)$

(i)
$$x(t)=1+2\delta(t-t_0)-\cos(w_0t)+\sum_{i=1}^{n}A_i e^{-a_i(t-t_i)}u(t-t_i)\ ,\ a_i\neq 1$$

$$\Rightarrow X(f)=\delta(f)+2e^{-j2\pi f t_0}-\frac{1}{2}\delta\left(f-\frac{w_0}{2\pi}\right)-\frac{1}{2}\delta\left(f+\frac{w_0}{2\pi}\right)+\sum_{i=1}^{n}A_i\frac{1}{a_i+j2\pi f}e^{-j2\pi f t_i}$$

$$\Rightarrow Y(f)=H(f)X(f)=\delta(f)\cdot\frac{1}{1+j2\pi 0}+\frac{2e^{-j2\pi f t_0}}{1+j2\pi f}-\frac{1}{2}\frac{1}{1+jw_0}\delta\left(f-\frac{w_0}{2\pi}\right)-\frac{1}{2}\frac{1}{1-jw_0}\delta\left(f+\frac{w_0}{2\pi}\right)$$
$$+\sum_{i=1}^{n}\frac{A_i e^{-j2\pi f t_i}}{(a_i+j2\pi f)(1+j2\pi f)}$$

$$=\delta(f)+\frac{2e^{-j2\pi f t_0}}{1+j2\pi f}-\frac{1}{2}\frac{1}{1+jw_0}\delta\left(f-\frac{w_0}{2\pi}\right)-\frac{1}{2}\frac{1}{1-jw_0}\delta\left(f+\frac{w_0}{2\pi}\right)$$
$$+\sum_{i=1}^{n}\left[\frac{\left(\frac{A_i}{1-a_i}\right)}{a_i+j2\pi f}+\frac{\left(\frac{-A_i}{1-a_i}\right)}{1+j2\pi f}\right]e^{-j2\pi f t_i}$$

$$\Rightarrow y(t)=1+2e^{-(t-t_0)}u(t-t_0)-\frac{1}{2}\frac{e^{jw_0t}}{1+jw_0}-\frac{1}{2}\frac{e^{-jw_0t}}{1-jw_0}+\sum_{i=1}^{n}\left[\left(\frac{A_i}{1-a_i}\right)e^{-a_i(t-t_i)}-\left(\frac{A_i}{1-a_i}\right)e^{-(t-t_i)}\right]u(t-t_i)$$
$$=1+2e^{-(t-t_0)}u(t-t_0)-\frac{\cos(w_0t)+w_0\sin(w_0t)}{1+w_0^2}+\sum_{i=1}^{n}\left(\frac{A_i}{1-a_i}\right)\left(e^{-a_i(t-t_i)}-e^{-(t-t_i)}\right)u(t-t_i)$$

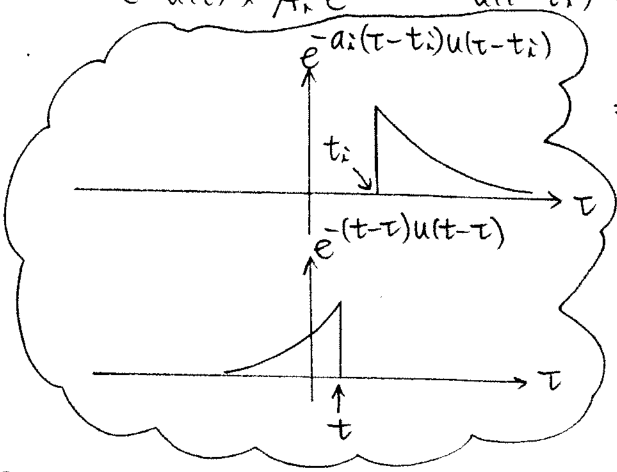
(ii) direct calculation :

① $e^{-t}u(t)*1=\int_{-\infty}^{\infty}e^{-\tau}u(\tau)1\,d\tau=\int_{0}^{\infty}e^{-\tau}d\tau=-\left[e^{-\infty}-e^{-0}\right]=1$

② $e^{-t}u(t)*2\delta(t-t_0)=\int_{0}^{\infty}e^{-\tau}2\delta(t-\tau-t_0)d\tau=2e^{-(t-t_0)}u(t-t_0)$

③ $e^{-t}u(t)*\cos(w_0t)=\frac{1}{2}\int_{0}^{\infty}e^{-\tau}e^{w_0(t-\tau)j}d\tau+\frac{1}{2}\int_{0}^{\infty}e^{-\tau}e^{-w_0(t-\tau)j}d\tau$
$$=\frac{1}{2}\frac{e^{jw_0t}}{1+jw_0}+\frac{1}{2}\frac{e^{-jw_0t}}{1-jw_0}=\frac{\cos(w_0t)+w_0\sin(w_0t)}{1+w_0^2}$$

④ $e^{-t}u(t)*A_ie^{-a_i(t-t_i)}u(t-t_i)=A_i\int_{-\infty}^{\infty}e^{-a_i(\tau-t_i)}u(\tau-t_i)e^{-(t-\tau)}u(t-\tau)d\tau$




$$=\begin{cases}A_i\int_{t_i}^{t}e^{-a_i(\tau-t_i)}\cdot e^{-(t-\tau)}d\tau\ , & t\geq t_i\\ 0\ , & t\leq t_i\end{cases}$$

$$=\begin{cases}\frac{A_i}{1-a_i}e^{-a_i(t-t_i)}-\frac{A_i}{1-a_i}e^{-(t-t_i)}\ , & t\geq t_i\\ 0\ , & t\leq t_i\end{cases}$$

$$=\left(\frac{A_i}{1-a_i}\right)\left(e^{-a_i(t-t_i)}-e^{-(t-t_i)}\right)u(t-t_i)$$

$$\xrightarrow{①②③④} y(t)=1+2e^{-(t-t_0)}u(t-t_0)-\frac{\cos(w_0t)+w_0\sin(w_0t)}{1+w_0^2}+\sum_{i=1}^{n}\left(\frac{A_i}{1-a_i}\right)\left(e^{-a_i(t-t_i)}-e^{-(t-t_i)}\right)u(t-t_i)$$

2. $\boxed{3.5\text{-}1}$

The impulse response $h(t)$ of a causal system has to be zero $\forall t<0$.

(a) Yes, $\because e^{-at}u(t)=0 \ , \forall t<0$.

(b) No. There exists $t_1<0$ such that $e^{-a|t_0|}\neq 0$

(c) Yes. $\because e^{-a(t-t_0)}u(t-t_0)=0 \ , \forall t<0 \quad (t_0\geq 0)$

(d) No. There exists $t_1<0$ such that $\operatorname{sinc}(at_1)\neq 0$.

(e) No. There exists $t_1<0$ such that $\operatorname{sinc}(a(t_1-t_0))\neq 0$.

3. $\boxed{3.5\text{-}2}$ $H(f)=e^{-(2\pi k|f|+j2\pi f t_0)}$

① $$h(t)=\int_{-\infty}^{\infty}H(f)e^{j2\pi ft}df=\int_0^{\infty}e^{-2\pi kf}\cdot e^{j2\pi f(t-t_0)}df+\int_{-\infty}^{0}e^{2\pi kf}\cdot e^{j2\pi f(t-t_0)}df$$

$$=\frac{1}{-2\pi k+j2\pi(t-t_0)}(-1)+\frac{1}{2\pi k+j2\pi(t-t_0)}=\frac{4\pi k}{4\pi^2k^2+4\pi^2(t-t_0)^2}$$

$$=\frac{k}{\pi k^2+\pi^2(t-t_0)^2}$$

$\Rightarrow h(t)$ can be zero for some $t<0 \Rightarrow$ noncausal and hence physically unrealizable. (Time-domain criterion)

A reasonable criterion: $h(0)=1\%$ of peak value which is $\frac{k}{\pi k^2}=\frac{1}{\pi k}$

i.e. $\frac{k}{\pi k^2+\pi^2 t_0^2}=0.01\cdot\frac{1}{\pi k} \Rightarrow t_0=k\sqrt{\frac{99}{\pi}}\approx 5.6136k$

This filter can be made approximately realizable by choosing a sufficiently large $t_0$.

② Frequency domain (Paley-Wiener) criterion:

$$\int_{-\infty}^{\infty}\frac{|\ln|H(f)||}{4\pi^2f^2+1}df=\int_{-\infty}^{\infty}\frac{|\ln e^{-(2\pi k|f|)}|}{4\pi^2f^2+1}df=\int_{-\infty}^{\infty}\frac{|2\pi kf|}{4\pi^2f^2+1}df=2\int_0^{\infty}\frac{2\pi kf}{4\pi^2f^2+1}df$$

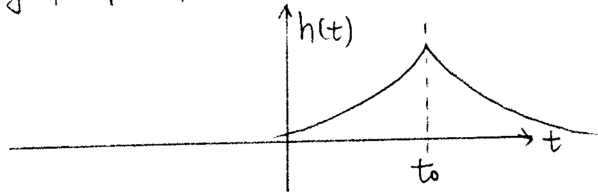
$$=2k\int_0^{\frac{\pi}{2}}\frac{\frac{\tan\theta}{2\pi}\cancel{\sec^2\theta}\,d\theta}{\cancel{\tan^2\theta+1}}=\frac{k}{\pi}\int_0^{\frac{\pi}{2}}\tan\theta\, d\theta=\frac{k}{\pi}\left(-\ln|\cos\theta|\Big|_0^{\frac{\pi}{2}}\right)=\infty \Rightarrow \text{physically unrealizable}$$

#

let $2\pi f=\tan\theta$, $2\pi df=\sec^2\theta\, d\theta$

4. $\boxed{3.5\text{-}3}$ $H(f)=\frac{2\beta}{(2\pi f)^2+\beta^2}e^{-j2\pi f t_0}$

By Table 3.1 and time-shifting property, $h(t)=e^{-\beta|t-t_0|}$, which is noncausal and hence unrealizable.




The exponential decays to 1.8% at 4 times constants. Hence $t_0=4\cdot\beta^{-1}$ is a reasonable choice to make this filter approximately realizable.

5. $\boxed{3.6\text{-}1}$ (a) $H(f)=e^{-j2\pi f t_0 - jk\sin 2\pi fT}$, $k \ll 1$

$$\approx e^{-j2\pi f t_0}[1-jk\sin(2\pi fT)] = e^{-j2\pi f t_0} - \frac{k}{2}e^{-j2\pi f(t_0-T)} + \frac{k}{2}e^{-j2\pi f(t_0+T)}$$

$\Rightarrow h(t) \approx \delta(t-t_0)+\frac{k}{2}\delta(t-t_0-T)-\frac{k}{2}\delta(t-t_0+T)$

$\Rightarrow$ Under this approximation, $y(t)=g(t-t_0)+\frac{k}{2}[g(t-t_0-T)-g(t-t_0+T)]$

(b) Under the approximation in (a), $h(t)$ is a linear combination of linear distortion impulse responses. According to 3.6 of textbook, The distortion resulting from $h(t)$ is undesirable in a TDM system since the pulse spreading causes interference with a neighboring pulse (i.e. Crosstalk)

For an FDM system, $h(t)$ causes distortion in each multiplexed signal while no interference occurs with a neighboring channel, because in FDM each multiplexed signal occupies a band which is not occupied by any other FDM signals.

6. $h_T(t) = \text{rect}\left(\frac{t-T/2}{T}\right) - \text{rect}\left(\frac{t+T/2}{T}\right)$

$$\Rightarrow H_T(f) = T\,\text{sinc}(\pi f T)\, e^{-j2\pi f T/2} - T\,\text{sinc}(\pi f T)\, e^{j2\pi f T/2}$$

$$= T\,\text{sinc}(\pi f T)(-2j)\sin(\pi f T)$$

$$= T\frac{\sin(\pi f T)}{\pi f T}(-2j)\sin(\pi f T)$$

$$= \frac{2}{j\pi f}\sin^2(\pi f T)$$

$$= \frac{2}{j\pi f}\cdot\frac{1}{2}\left[1-\cos(2\pi f T)\right] = \frac{1}{j\pi f} - \frac{\cos(2\pi f T)}{j\pi f} \neq \frac{1}{j\pi f}$$

We observe $H_T(f)$ cannot be determined as $T \to \infty$.

We can express $h_T(t)$ in another way :

$$h_T(t) = \text{rect}\left(\frac{t-T/2}{T}\right) - \text{rect}\left(\frac{t+T/2}{T}\right)$$

$$= \frac{1}{2}\left(\text{sgn}(t) - \text{sgn}(t-T)\right) - \frac{1}{2}\left(\text{sgn}(t+T) - \text{sgn}(t)\right)$$

$$= \text{sgn}(t) - \frac{1}{2}\underbrace{\left(\text{sgn}(t-T) + \text{sgn}(t+T)\right)}_{\neq 0 \text{ for all positive } T}$$

$$\neq \text{sgn}(t), \ \forall\, T>0$$

Hence $H_T(f) \neq \frac{1}{j\pi f}$, $\forall\, T>0$

7. $\boxed{3.6-3}$ $y(t) = x(t) + 0.001\, x^2(t)$

(a) $x(t) = \frac{1000}{\pi} \operatorname{sinc}(1000t) \Rightarrow y(t) = \frac{1000}{\pi}\operatorname{sinc}(1000t) + 0.001\left(\frac{1000}{\pi}\right)^2 \operatorname{sinc}^2(1000t)$

$$= \frac{1000}{\pi}\operatorname{sinc}(1000t) + \frac{1000}{\pi^2}\operatorname{sinc}^2(1000t)$$

By Fourier Transform Pairs : $\operatorname{sinc}(\pi t) \rightleftharpoons \operatorname{rect}(f)$

$\operatorname{sinc}^2(\pi t) \rightleftharpoons \Delta\left(\frac{f}{2}\right)$

We have : $Y(f) = \frac{1000}{\pi}\cdot\frac{\pi}{1000}\operatorname{rect}\left(\frac{f}{\left(\frac{1000}{\pi}\right)}\right) + 0.001\cdot\left(\frac{1000}{\pi}\right)^2\cdot\frac{\pi}{1000}\Delta\left(\frac{f}{2\,\frac{1000}{\pi}}\right)$

$$= \operatorname{rect}\left(\frac{f}{\left(\frac{1000}{\pi}\right)}\right) + \frac{1}{\pi}\Delta\left(\frac{f}{2\left(\frac{1000}{\pi}\right)}\right)$$

(b) From (a), the bandwidth of $x(t)$ is $\frac{500}{\pi}$ and the bandwidth of $y(t)$ is $\frac{1000}{\pi}$. Thus that the bandwidth of the output signal is twice that of the input signal is verified.

(c) For general $x(t)$, it cannot be recovered without distortion since the signal squaring introduces nonlinear distortion that occupies the bandwidth which is overlapped with the bandwidth of original $x(t)$.

8. $\boxed{3.7-1}$ We know $\operatorname{sinc}^2(\pi t) \rightleftharpoons \Delta\left(\frac{f}{2}\right)$ Hence, $\operatorname{sinc}^2(kt) = \operatorname{sinc}^2\left(\pi t \frac{k}{\pi}\right) \rightleftharpoons \frac{\pi}{k}\Delta\left(\frac{f}{2\left(\frac{k}{\pi}\right)}\right)$

$$\int_{-\infty}^{\infty} \operatorname{sinc}^4(kt)\,dt = \int_{-\infty}^{\infty}\left|\operatorname{sinc}^2(kt)\right|^2 dt \underset{\text{Parseval's Theorem}}{=} \int_{-\infty}^{\infty}\left|\frac{\pi}{k}\Delta\left(\frac{f}{2\left(\frac{k}{\pi}\right)}\right)\right|^2 df = \frac{\pi^2}{k^2}\int_{-\frac{k}{\pi}}^{\frac{k}{\pi}}\left|1-|f|\frac{\pi}{k}\right|^2 df$$

$$= \frac{2\pi^2}{k^2}\int_0^{\frac{k}{\pi}}\left(1-f\frac{\pi}{k}\right)^2 df = \frac{2\pi^2}{k^2}\int_0^{\frac{k}{\pi}}\left(1-2f\frac{\pi}{k}+f^2\frac{\pi^2}{k^2}\right)df = \frac{2\pi^2}{k^2}\left(f-\frac{\pi}{k}f^2+\frac{1}{3}\frac{\pi^2}{k^2}f^3\Big|_0^{\frac{k}{\pi}}\right)$$

$$= \frac{2\pi^2}{k^2}\left(\frac{k}{\pi}-\frac{\pi}{k}\frac{k^2}{\pi^2}+\frac{1}{3}\frac{\pi^2}{k^2}\cdot\frac{k^3}{\pi^3}\right)$$

$$= \frac{2\pi}{3k}$$

9. $\boxed{3.7\text{-}4}$ $g(t) = \frac{2a}{t^2+a^2}$

From table 3.1, we know $x(t) = e^{-a|t|},\ a>0 \rightleftharpoons X(f) = \frac{2a}{a^2+(2\pi f)^2}$

By duality property, $X(t) \rightleftharpoons x(-f)$, i.e., $\frac{2a}{a^2+(2\pi t)^2} \rightleftharpoons e^{-a|-f|} = e^{-a|f|}$, $a>0$

$$\Rightarrow \frac{2a}{t^2+a^2} = \frac{2a}{(\frac{t}{2\pi}\cdot 2\pi)^2+a^2} \rightleftharpoons 2\pi e^{-a|2\pi f|},\ a>0$$

$$\Rightarrow E_g = \int_{-\infty}^{\infty}|G(f)|^2 df = 4\pi^2\int_{-\infty}^{\infty} e^{-2a|2\pi f|}df = 4\pi^2\int_{-\infty}^{\infty} e^{-4\pi a|f|}df = 8\pi^2\int_0^{\infty} e^{-4\pi a f}df$$

$$= 8\pi^2\cdot\frac{1}{4\pi a} = \frac{2\pi}{a}.$$

$$E_{g,B} = \int_{-B}^{B}|G(f)|^2 df = 8\pi^2\int_0^{B} e^{-4\pi a f}df = \frac{2\pi}{a}\left[1-e^{-4\pi a B}\right] = \frac{2\pi}{a}\times 0.99$$

$$\Rightarrow e^{-4\pi a B} = 0.01 \Rightarrow B = \frac{1}{4\pi a}\ln 100 \simeq \frac{0.3665}{a}\ \text{Hz}$$

10. $\boxed{3.7\text{-}5}$

Let $g^2(t) \rightleftharpoons A(f)$, $y(t) \rightleftharpoons Y(f)$. We have $Y(f) = A(f)\,\mathrm{rect}(\frac{f}{2\Delta f})$

Because $\mathrm{sinc}(\pi t) \rightleftharpoons \mathrm{rect}(f)$

$$\Rightarrow \mathrm{sinc}(\pi t 2\Delta f) \rightleftharpoons \frac{1}{2\Delta f}\mathrm{rect}(\frac{f}{2\Delta f})$$

$$\Rightarrow 2\Delta f\,\mathrm{sinc}(\pi t 2\Delta f) \rightleftharpoons \mathrm{rect}(\frac{f}{2\Delta f})$$

When $\Delta f$ is very small, $Y(f) \approx A(0)\,\mathrm{rect}(\frac{f}{2\Delta f})$

Hence $y(t) \approx A(0)\,2\Delta f\,\mathrm{sinc}(\pi t 2\Delta f)$

$$= A(0)\,2\Delta f\,\frac{\sin(\pi t 2\Delta f)}{\pi t 2\Delta f} \approx A(0)\,2\Delta f\,\frac{\cancel{\pi t 2\Delta f}}{\cancel{\pi t 2\Delta f}} = A(0)2\Delta f$$

$\sin(x) \approx x$ when $x$ is very small

On the other hand,

$$A(f) = \int_{-\infty}^{\infty} g^2(t)e^{-j2\pi f t}dt \Rightarrow A(0) = \int_{-\infty}^{\infty} g^2(t)dt = E_g$$

Therefore $y(t) \approx A(0)\,2\Delta f = 2E_g\Delta f$ ※