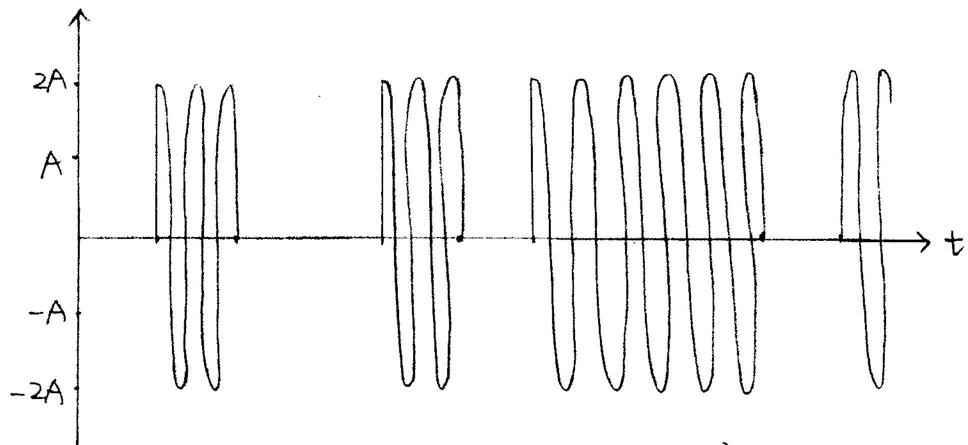
HW5 Solutions

1. $\boxed{4.3\text{-}7}$

$\varphi_{AM}(t) = (A + m(t))\cos(2\pi f_c t)$, m(t) is shown in Fig. P4.3-7 with $m_p = A$, i.e., $\mu = 1$.



$$\varphi_{AM}(t) = \begin{cases} 2A\cos(2\pi f_c t), & \text{if } m(t) = A \\ 0, & \text{if } m(t) = -A \end{cases}$$

$$\varphi_{AM}(t) = \underbrace{A\cos(2\pi f_c t)}_{\text{carrier}} + \underbrace{m(t)\cos(2\pi f_c t)}_{\text{sidebands}}$$

According to P. 194 of textbook (or see Eq. (3.94)), $P_c = \frac{A^2}{2}$. $P_s = \frac{1}{2}\overline{m^2(t)} = \frac{A^2}{2}$

$\Rightarrow$ sideband's power $= P_s = \frac{A^2}{2}$, total power $= P_c + P_s = A^2$,

$$\Rightarrow \eta = \frac{P_s}{P_c + P_s} = \frac{\left(\frac{A^2}{2}\right)}{A^2} = \frac{1}{2} = 50\%$$

2. $\boxed{4.3\text{-}8}$

ⓐ $\varphi_{AM}(t) = (A + m(t))\cos(2\pi f_c t)$

ⓑ $x(t) = \varphi_{AM}^2(t) = (A + m(t))^2\cos^2(2\pi f_c t) = \frac{1}{2}(A + m(t))^2 + \frac{1}{2}(A + m(t))^2\cos(4\pi f_c t)$

ⓒ The output of Low-pass filter is $\frac{1}{2}(A + m(t))^2 = \frac{1}{2}A^2 + Am(t) + \frac{1}{2}m^2(t)$

ⓓ $y(t) = Am(t) + \frac{1}{2}m^2(t)$, where $\frac{1}{2}m^2(t)$ is distortion component.

$= m(t)\left(A + \frac{1}{2}m(t)\right)$

$\approx Am(t)$, which is proportional to m(t) (i.e., scaled version of m(t))

$(A \gg |m(t)|)$ Hence if $A \gg |m(t)|$, the distortion is small, compared to $Am(t)$.

3. $\boxed{4.3\text{-}9}$

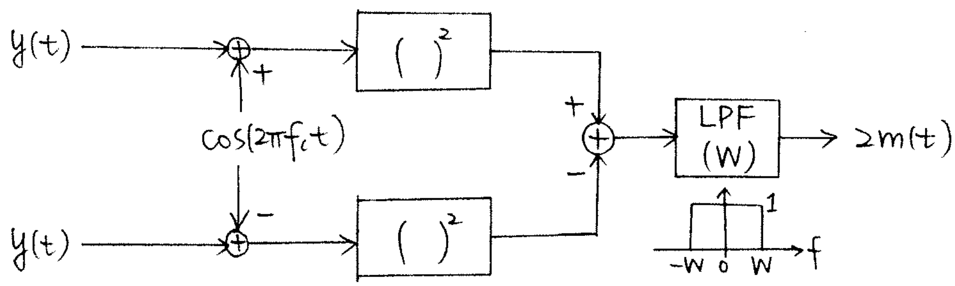
Let $y(t) = m(t)\cos(2\pi f_c t)$, assume $m \in LP(W)$.

$$\varphi_1(t) \triangleq \left(y(t)+\cos(2\pi f_c t)\right)^2 = m^2(t)\cos^2(2\pi f_c t) + 2m(t)\cos^2(2\pi f_c t) + \cos^2(2\pi f_c t)$$

$$\varphi_2(t) \triangleq \left(y(t)-\cos(2\pi f_c t)\right)^2 = m^2(t)\cos^2(2\pi f_c t) - 2m(t)\cos^2(2\pi f_c t) + \cos^2(2\pi f_c t)$$

$$\Rightarrow \varphi_1(t)-\varphi_2(t) = 4m(t)\cos^2(2\pi f_c t) = 2m(t) + 2\cos(2\pi(2f_c)t)$$

and $2m(t)$ can be obtained by passing $\varphi_1(t)-\varphi_2(t)$ through a LPF with bandwidth W



4. $\boxed{4.4\text{-}1}$ Use notations in Figure 4.19:

$\varphi_{QAM}(t) = m_1(t)\cos(\omega_c t) + m_2(t)\sin(\omega_c t)$,

If local carrier has frequency error $\Delta\omega$ and phase error δ, then

$$\begin{aligned}
x_1(t) &= 2\varphi_{QAM}(t)\cos((\omega_c+\Delta\omega)t+\delta) \\
&= 2m_1(t)\cos(\omega_c t+\Delta\omega t+\delta)\cos(\omega_c t) + 2m_2(t)\cos(\omega_c t+\Delta\omega t+\delta)\sin(\omega_c t) \\
&= m_1(t)\cos(2\omega_c t+\Delta\omega t+\delta) + m_1(t)\cos(\Delta\omega t+\delta) \\
&\quad + m_2(t)\sin(2\omega_c t+\Delta\omega t+\delta) - m_2(t)\sin(\Delta\omega t+\delta),
\end{aligned}$$

$$\begin{aligned}
x_2(t) &= 2\varphi_{QAM}(t)\sin((\omega_c+\Delta\omega)t+\delta) \\
&= 2m_1(t)\sin(\omega_c t+\Delta\omega t+\delta)\cos(\omega_c t) + 2m_2(t)\sin(\omega_c t+\Delta\omega t+\delta)\sin(\omega_c t) \\
&= m_1(t)\sin(2\omega_c t+\Delta\omega t+\delta) + m_1(t)\sin(\Delta\omega t+\delta) \\
&\quad + m_2(t)\cos(\Delta\omega t+\delta) - m_2(t)\cos(2\omega_c t+\Delta\omega t+\delta).
\end{aligned}$$

Therefore, the output of upper branch is $m_1(t)\cos(\Delta\omega t+\delta) - m_2(t)\sin(\Delta\omega t+\delta)$,

and the output of lower branch is $m_1(t)\sin(\Delta\omega t+\delta) + m_2(t)\cos(\Delta\omega t+\delta)$

※

5.

$$G(f) = \int_{\mathbb{R}} g(t) e^{-j2\pi ft} dt$$

$$\Rightarrow |G(f)| \le \int_{\mathbb{R}} |g(t)| dt < \infty \Rightarrow G(f) \text{ is well defined}$$

$$\Rightarrow -j\,\mathrm{sgn}(f)\, G(f) = G_h(f) \text{ is well defined}$$

Similarly, since $g_h(t) = \int_{\mathbb{R}} G_h(f) e^{j2\pi ft} df$, we have $|g_h(t)| \le \int_{\mathbb{R}} |G_h(f)| df$

$$= \int_{\mathbb{R}} |-j\,\mathrm{sgn}(f)\, G(f)| df$$

$$= \int_{\mathbb{R}} |G(f)| df < \infty$$

Hence, $g_h(t)$ is well defined.

Therefore, whenever $\int_{\mathbb{R}} |G(f)| df < \infty$, we have $g_h(t)$ well defined.

Also, $\int_{\mathbb{R}} |g_h(t)|^2 dt = \int_{\mathbb{R}} |G_h(f)|^2 df = \int_{\mathbb{R}} |-j\,\mathrm{sgn}(f)\, G(f)|^2 df = \int_{\mathbb{R}} |G(f)|^2 df$

$$= \int_{\mathbb{R}} |g(t)|^2 dt < \infty.$$

Furthermore,

$\int_{\mathbb{R}} g(t) g_h(t) dt = Y(0)$, where $Y(f) = G * G_h(f)$

$$= \int_{-\infty}^{\infty} G(f-v) G_h(v) dv$$

$$= \int_{-\infty}^{\infty} G(f-v)(-j)\,\mathrm{sgn}(v) G(v)\, dv$$

$$= (-j) \int_{-\infty}^{\infty} G(f-v)\,\mathrm{sgn}(v) G(v) dv$$

$$\Rightarrow Y(0) = (-j)\left[-\int_{-\infty}^{0} G(-v) G(v) dv + \int_{0}^{\infty} G(-v) G(v) dv \right]$$

$$= (-j)\left[\int_{\infty}^{0} G(f) G(-f) df + \int_{0}^{\infty} G(-v) G(v) dv \right]$$

$$= (-j)\left[-\int_{0}^{\infty} G(f) G(-f) df + \int_{0}^{\infty} G(-v) G(v) dv \right]$$

$$= (-j) \cdot 0$$

$$= 0$$

Thus, $\int_{\mathbb{R}} g(t) g_h(t) dt = 0$, g and g_h are orthogonal signals. #

6. $\boxed{4.4\text{-}3}$

The Hilbert transformer is an ideal phase shifter that shifts the phase of every positive spectral component by $-\frac{\pi}{2}$ (see p. 199–200 of textbook)

(a) $m(t) = \cos(100\pi t) + 2\cos(300\pi t) \Rightarrow m_h(t) = \cos(100\pi t - \frac{\pi}{2}) + 2\cos(300\pi t - \frac{\pi}{2})$

$$\varphi_{USB}(t) = \left[\cos(100\pi t) + 2\cos(300\pi t)\right]\cos(1000\pi t) - \left[\cos(100\pi t - \tfrac{\pi}{2}) + 2\cos(300\pi t - \tfrac{\pi}{2})\right]\sin(1000\pi t)$$

$$= \left[\cos(100\pi t) + 2\cos(300\pi t)\right]\cos(1000\pi t) - \left[\sin(100\pi t) + 2\sin(300\pi t)\right]\sin(1000\pi t)$$

$$= \tfrac{1}{2}\cos(1100\pi t) + \cancel{\tfrac{1}{2}\cos(900\pi t)} + \cos(1300\pi t) + \cancel{\cos(700\pi t)}$$

$$+ \tfrac{1}{2}\cos(1100\pi t) - \cancel{\tfrac{1}{2}\cos(900\pi t)} + \cos(1300\pi t) - \cancel{\cos(700\pi t)}$$

$$= \cos(1100\pi t) + 2\cos(1300\pi t)$$

$$\varphi_{LSB}(t) = \left[\cos(100\pi t) + 2\cos(300\pi t)\right]\cos(1000\pi t) + \left[\cos(100\pi t - \tfrac{\pi}{2}) + 2\cos(300\pi t - \tfrac{\pi}{2})\right]\sin(1000\pi t)$$

$$= \cancel{\tfrac{1}{2}\cos(1100\pi t)} + \tfrac{1}{2}\cos(900\pi t) + \cancel{\cos(1300\pi t)} + \cos(700\pi t)$$

$$- \cancel{\tfrac{1}{2}\cos(1100\pi t)} + \tfrac{1}{2}\cos(900\pi t) - \cancel{\cos(1300\pi t)} + \cos(700\pi t)$$

$$= \cos(900\pi t) + 2\cos(700\pi t)$$

(b) $m(t) = \sin(100\pi t)\sin(500\pi t) = \frac{1}{2}\cos(400\pi t) - \frac{1}{2}\cos(600\pi t)$

$\Rightarrow m_h(t) = \frac{1}{2}\cos(400\pi t - \frac{\pi}{2}) - \frac{1}{2}\cos(600\pi t - \frac{\pi}{2}) = \frac{1}{2}\sin(400\pi t) - \frac{1}{2}\sin(600\pi t)$

$$\varphi_{USB}(t) = \tfrac{1}{2}\left[\cos(400\pi t) - \cos(600\pi t)\right]\cos(1000\pi t) + \tfrac{1}{2}\left[\sin(600\pi t) - \sin(400\pi t)\right]\sin(1000\pi t)$$

$$= \tfrac{1}{4}\cos(1400\pi t) + \cancel{\tfrac{1}{4}\cos(600\pi t)} - \tfrac{1}{4}\cos(1600\pi t) - \cancel{\tfrac{1}{4}\cos(400\pi t)}$$

$$+ \cancel{\tfrac{1}{4}\cos(400\pi t)} - \tfrac{1}{4}\cos(1600\pi t) + \tfrac{1}{4}\cos(1400\pi t) - \cancel{\tfrac{1}{4}\cos(600\pi t)}$$

$$= \tfrac{1}{2}\cos(1400\pi t) - \tfrac{1}{2}\cos(1600\pi t)$$

$$\varphi_{LSB}(t) = \tfrac{1}{2}\left[\cos(400\pi t) - \cos(600\pi t)\right]\cos(1000\pi t) - \tfrac{1}{2}\left[\sin(600\pi t) - \sin(400\pi t)\right]\sin(1000\pi t)$$

$$= \cancel{\tfrac{1}{4}\cos(1400\pi t)} + \tfrac{1}{4}\cos(600\pi t) - \cancel{\tfrac{1}{4}\cos(1600\pi t)} - \tfrac{1}{4}\cos(400\pi t)$$

$$- \tfrac{1}{4}\cos(400\pi t) + \cancel{\tfrac{1}{4}\cos(1600\pi t)} - \cancel{\tfrac{1}{4}\cos(1400\pi t)} + \tfrac{1}{4}\cos(600\pi t)$$

$$= \tfrac{1}{2}\cos(600\pi t) - \tfrac{1}{2}\cos(400\pi t)$$

%.

$$g_\theta(t) = \int_{\mathbb{R}} G_\theta(f) e^{j2\pi ft} df$$

$$\Rightarrow |g_\theta(t)| \le \int_{\mathbb{R}} |G_\theta(f)| df = \int_{\mathbb{R}} |H_\theta(f) G(f)| df = \int_{\mathbb{R}} |H_\theta(f)||G(f)| df = \int_{\mathbb{R}} |G(f)| df$$

Hence, whenever $\int_{\mathbb{R}} |G(f)| df < \infty$, $g_\theta(t)$ is well defined.

(we assume $\int_{\mathbb{R}} |g(t)| dt < \infty$)

$$G_\theta(f) = H_\theta(f) G(f), \quad f \in \mathbb{R}$$

$$= \begin{cases} G(f) e^{-j\theta}, & f > 0 \\ 0, & f = 0 \\ G(f) e^{j\theta}, & f < 0 \end{cases}$$

$$= \begin{cases} G(f)\cos\theta - jG(f)\sin\theta, & f > 0 \\ 0, & f = 0 \\ G(f)\cos\theta + jG(f)\sin\theta, & f < 0 \end{cases}$$

$$= \begin{cases} G(f)\cos\theta + G_h(f)\sin\theta, & f > 0 \\ 0, & f = 0 \\ G(f)\cos\theta + G_h(f)\sin\theta, & f < 0 \end{cases}, \quad \text{where } G_h(f) = H(f)G(f)$$

$$= \begin{cases} G(f)\cos\theta + G_h(f)\sin\theta, & f \neq 0 \\ 0, & f = 0 \end{cases}$$

We have $g_\theta(t) = \int_{\mathbb{R}} G_\theta(f) e^{j2\pi ft} df = \int_{\mathbb{R}} [G(f)\cos\theta + G_h(f)\sin\theta] e^{j2\pi ft} df$

$$= \cos\theta \int_{\mathbb{R}} G(f) e^{j2\pi ft} df + \sin\theta \int_{\mathbb{R}} G_h(f) e^{j2\pi ft} df$$

$$= g(t)\cos\theta + g_h(t)\sin\theta$$

Thus, g_θ is a linear combination of g and g_h ※

8. $\boxed{4.4\text{-}5}$ (a) $\mathcal{H}(m(t)) = m_h(t)$, $M_h(f) = (-j\,\mathrm{sgn}(f))M(f)$

$\Rightarrow \mathcal{F}(\mathcal{H}(m_h(t))) = (-j\,\mathrm{sgn}(f))M_h(f) = (-j)\mathrm{sgn}(f)(-j)\mathrm{sgn}(f)M(f) = -M(f)$

$\Rightarrow \mathcal{H}(m_h(t)) = -m(t)$

(b) $$E_m = \int_{-\infty}^{\infty} |m(t)|^2 dt = \int_{-\infty}^{\infty} |M(f)|^2 df = \int_{-\infty}^{\infty} |-j\,\mathrm{sgn}(f)M(f)|^2 df = \int_{-\infty}^{\infty} |M_h(f)|^2 df$$

$$= \int_{-\infty}^{\infty} |m_h(t)|^2 dt = E_{m_h} \;\#$$

9. $\boxed{4.4\text{-}6}$

According to problem statement, $\varphi_{LSB}(t) = 2\cos[(w_c+\Delta w)t+\delta]m(t) + 2\sin[(w_c+\Delta w)t+\delta]m_h(t)$

$\Rightarrow \varphi_{LSB}(t)\cos(w_c t) = m(t)\cos((2w_c+\Delta w)t+\delta) + m(t)\cos(\Delta w t+\delta)$

$+ m_h(t)\sin((2w_c+\Delta w)t+\delta) + m_h(t)\sin(\Delta w t+\delta)$

Passing $\varphi_{LSB}\cos(w_c t)$ through a LPF, we have $y(t) = m(t)\cos(\Delta w t+\delta) + m_h(t)\sin(\Delta w t+\delta)$

(a) When $\delta=0$, $y(t) = m(t)\cos(\Delta w t) + m_h(t)\sin(\Delta w t)$, which is a LSB signal corresponding to a small carrier frequency Δw. This changes the sound of an audio signal slightly. For voice signals, the frequency shift within ± 20Hz is considered tolerable. Most US systems restrict the shift to ± 2Hz though.

(b) When $\Delta w = 0$, $y(t) = m(t)\cos(\delta) + m_h(t)\sin(\delta)$

$\Rightarrow Y(f) = M(f)\cos(\delta) - j\,\mathrm{sgn}(f)M(f)\sin(\delta)$

$$= \begin{cases} M(f)\cos(\delta) - jM(f)\sin(\delta) = M(f)e^{-j\delta}, & \text{if } f>0 \\ M(f)\cos(\delta) + jM(f)\sin(\delta) = M(f)e^{j\delta}, & \text{if } f<0 \end{cases}$$

Hence the output is the signal $m(t)$ with phases of all its spectral components shifted by δ. For voice signals, phase distortion is generally not a series problem because the human ear is somewhat insensitive to phase distortion, which may change the quality of speech while the voice is still intelligible.

10. $\boxed{4.4\text{-}7}$

From $\boxed{4.4\text{-}5}$ we know the Hilbert transform of $m_h(t)$ is $-m(t)$.

If $m_h(t)$ is the input, the USB output is

$$\tilde{\varphi}_{USB}(t) = m_h(t)\cos(\omega_c t) - (-m(t))\sin(\omega_c t)$$

$$= m(t)\sin(\omega_c t) + m_h(t)\cos(\omega_c t)$$

$$= m(t)\cos(\omega_c t - \tfrac{\pi}{2}) - m_h(t)\sin(\omega_c t - \tfrac{\pi}{2}),$$

and the LSB output is $\tilde{\varphi}_{LSB}(t) = m_h(t)\cos(\omega_c t) + (-m(t))\sin(\omega_c t)$

$$= m(t)(-\sin(\omega_c t)) + m_h(t)\cos(\omega_c t)$$

$$= m(t)\cos(\omega_c t + \tfrac{\pi}{2}) + m_h(t)\sin(\omega_c t + \tfrac{\pi}{2})$$

Therefore, $\tilde{\varphi}_{USB}(t)$ is an USB signal corresponding to $m(t)$, with phase shift $-\frac{\pi}{2}$ in the carrier; $\tilde{\varphi}_{LSB}(t)$ is a LSB signal corresponding to $m(t)$, with phase shift $\frac{\pi}{2}$ in the carrier.

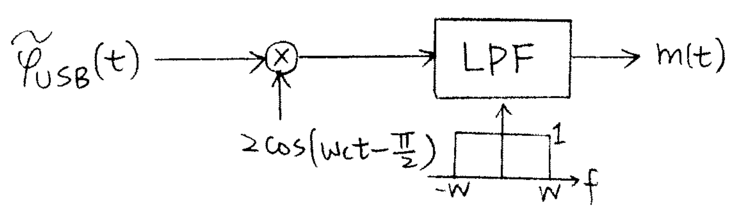
Both $\tilde{\varphi}_{USB}(t)$ and $\tilde{\varphi}_{LSB}(t)$ are SSB signals with bandwidth equal to that of $m(t)$. They can be demodulated to get back $m(t)$.

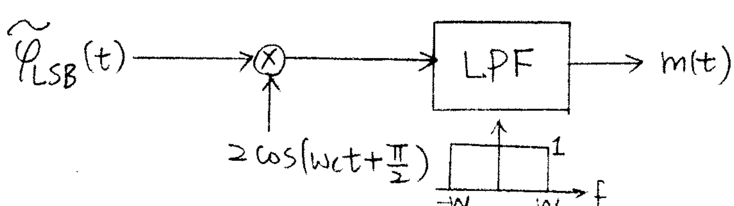
$$\tilde{y}_{USB}(t) \triangleq \tilde{\varphi}_{USB}(t)\, 2\cos(\omega_c t - \tfrac{\pi}{2}) = m(t)\cos(2\omega_c t - \pi) + m(t) - m_h(t)\sin(2\omega_c t - \pi)$$

$\Rightarrow$ $m(t)$ is obtained by passing $\tilde{y}_{USB}(t)$ through an LPF.

$$\tilde{y}_{LSB}(t) \triangleq \tilde{\varphi}_{LSB}(t)\, 2\cos(\omega_c t + \tfrac{\pi}{2}) = m(t)\cos(2\omega_c t + \pi) + m(t) + m_h(t)\sin(2\omega_c t + \pi)$$

$\Rightarrow$ $m(t)$ is obtained by passing $\tilde{y}_{LSB}(t)$ through an LPF.





Assume $m(t) \in LP(W)$.