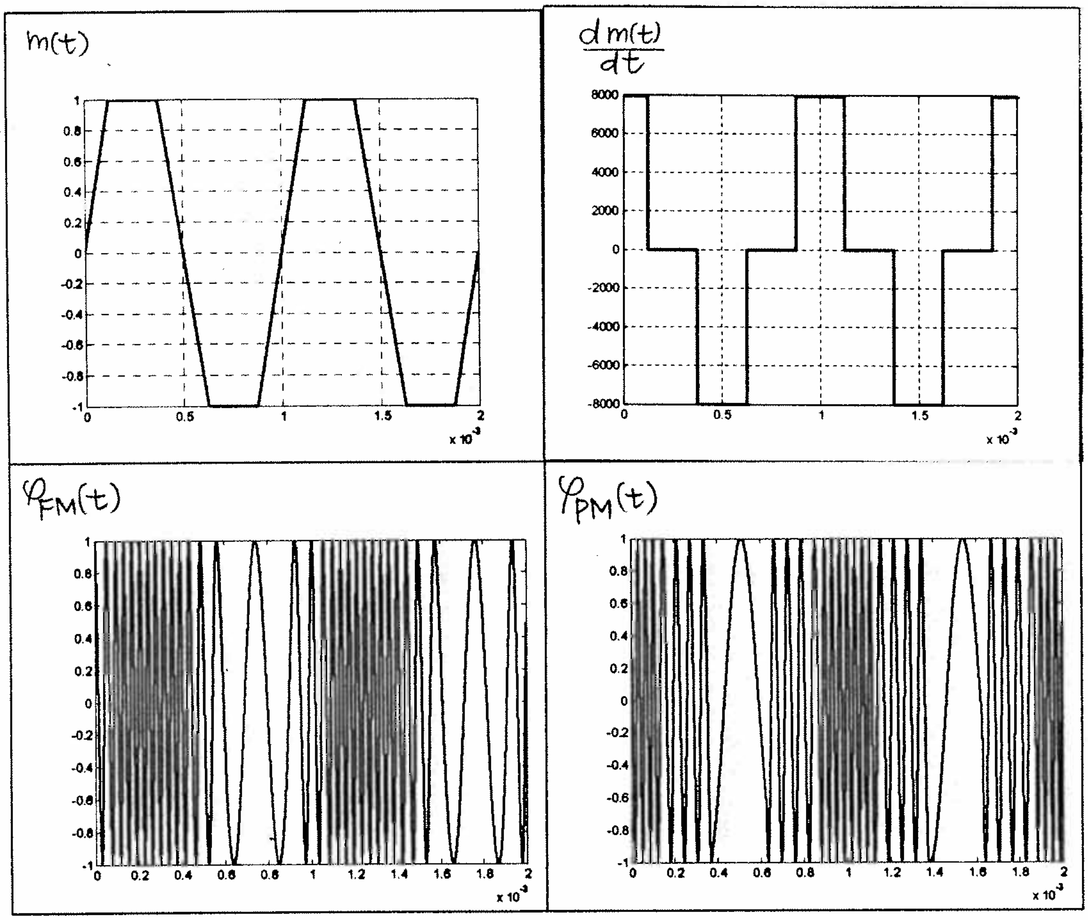
HW #6 Solutions

1. $\boxed{5.1\text{-}1}$

$\omega_c = 10^8,\ k_f = 10^5,\ k_p = 25.$



$$\varphi_{FM}(t) = A_c \cos\left(\omega_c t + k_f \int_{-\infty}^{t} m(\tau)\, d\tau + \theta_0\right) \qquad (5.5)^*$$

$$\varphi_{PM}(t) = A_c \cos\left(\omega_c t + k_p m(t) + \theta_0\right) \qquad (5.3b)^*$$

we pick $A_c = 1$ and $\theta_0 = 0$.

For $\varphi_{FM}(t)$, the instantaneous frequency $\omega_i = 10^8 + 10^5 m(t)$ (rad/s)

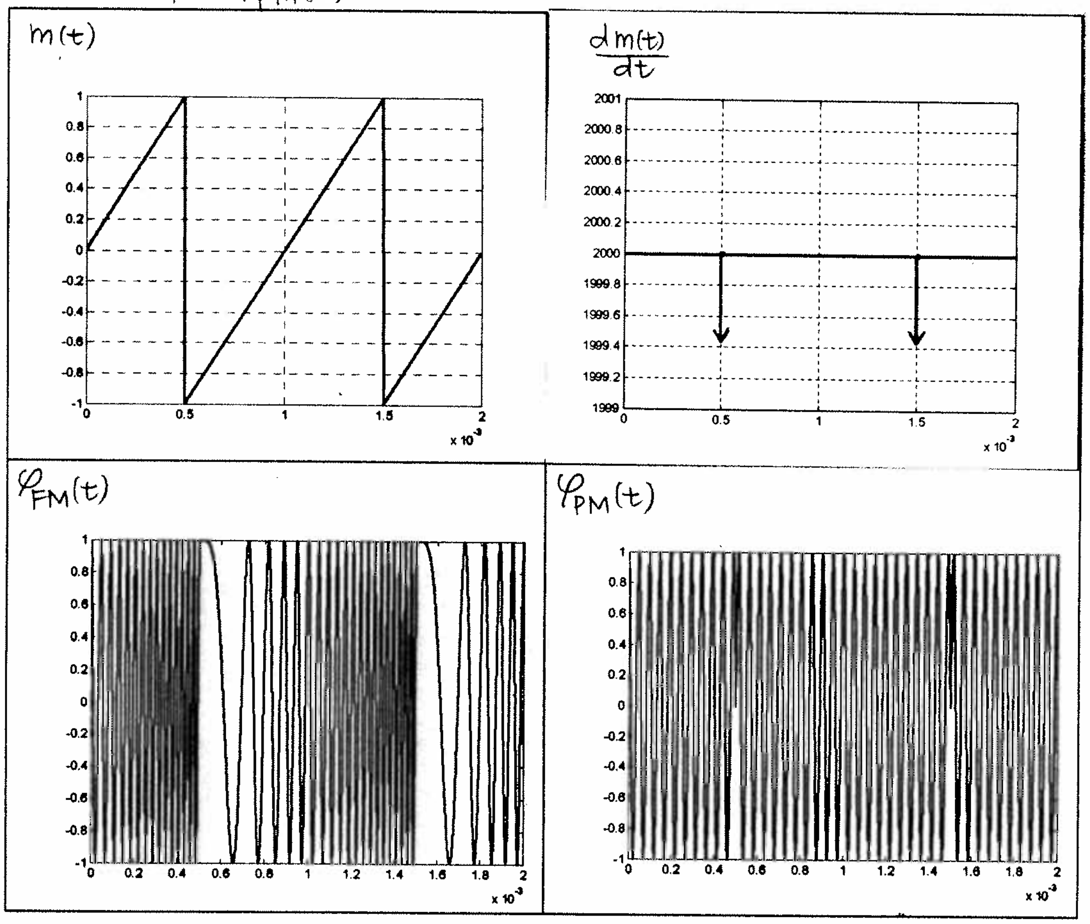
Since $m(t) \in [1, -1]$, $\omega_i \in [10^8 - 10^5,\ 10^8 + 10^5] \cong [9.99 \times 10^7,\ 1.001 \times 10^8]$ (rad/s)

For $\varphi_{PM}(t)$, $\omega_i = 10^8 + 25 \frac{dm(t)}{dt}$. Since $\frac{dm(t)}{dt} \in [-8000, 8000]$, we have $\omega_i \in [10^8 - 2\times10^5,\ 10^8 + 2\times 10^5] \simeq [9.98 \times 10^7,\ 1.002 \times 10^8]$ (rad/s)

* Here k_f and k_p follow definitions in the textbook.
($k_f = 2\pi k_F$)

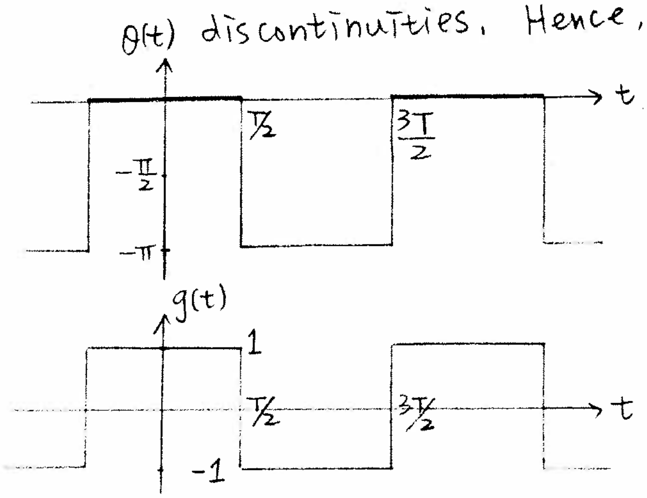
2. $\boxed{5.1\text{-}2}$ $\omega_c = 2\pi \times 10^6$, $k_f = 2000\pi$, $k_p = \frac{\pi}{2}$

(a) See sketches below. For $\varphi_{FM}(t)$, instantaneous frequency $f_i = 10^6 + 1000\,m(t)$ Hz
$\Rightarrow f_i \in [999k, 1001k]$ Hz.
See (b) for $\varphi_{PM}(t)$ case.



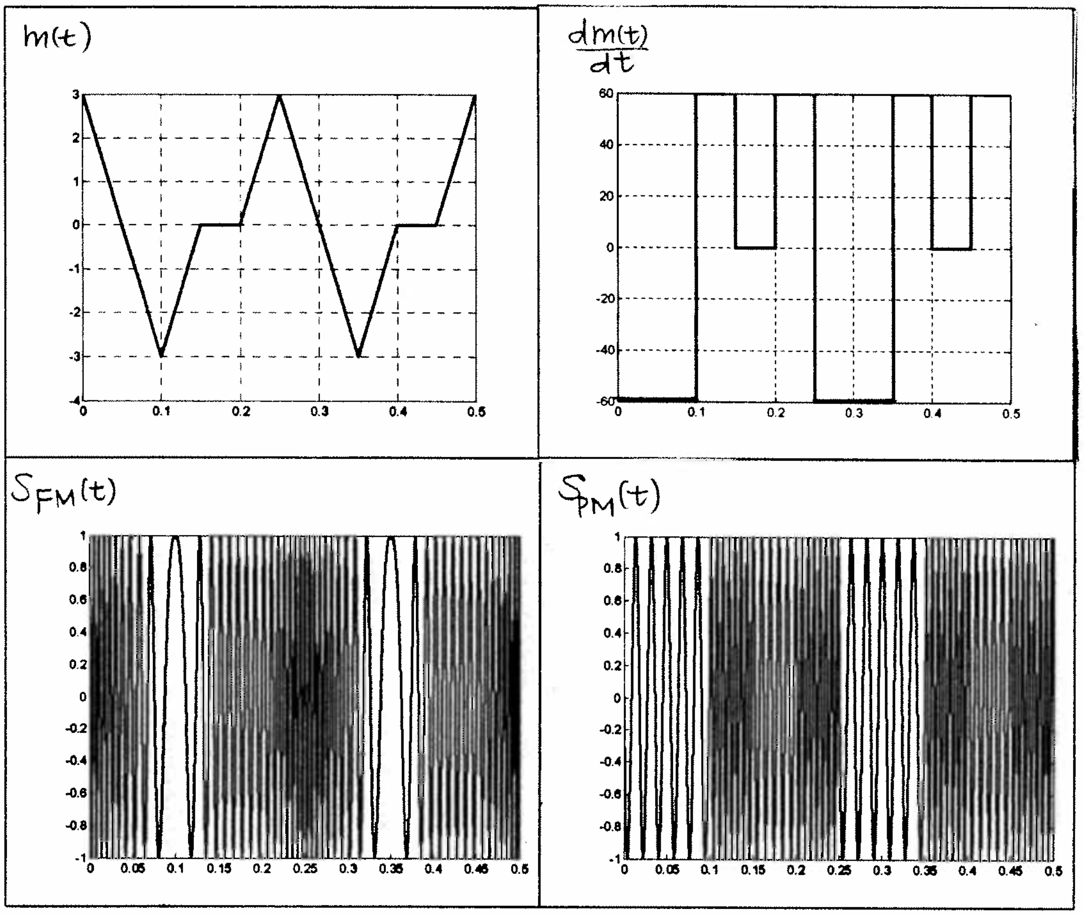
(b) Without loss of generality, we assume one cycle of the sawtooth of $m(t)$ is centered at the origin. Over that cycle, $m(t) = 2000t$ and
$\varphi_{PM}(t) = A_c \cos\left(2\pi \times 10^6 t + \frac{\pi}{2} \times 2000t\right) = A_c \cos\left(2\pi(10^6 + 500)t\right)$.
At discontinuitis of $m(t)$, i.e., $t = \frac{T}{2}, \frac{3}{2}T, \frac{5}{2}T, \cdots$, there is a phase jump from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$, which means there is a phase shift $-\pi$ at these discontinuities. Hence, $\varphi_{PM}(t) = A_c \cos\left(2\pi \times (10^6 + 500)t + \theta(t)\right)$

$= A_c \cos\left(2\pi \times (10^6 + 500)t + \frac{\pi}{2} g(t) - \frac{\pi}{2}\right)$, where $\theta(t)$ and $g(t)$ are shown at the right.
$g(t)$ is a periodic rectangular pulse signal
$\Rightarrow \varphi_{PM}(t)$ is equivalent to another PM signal with $f_c = 10^6 + 500$ Hz modulated by $g(t)$



Furthermore, we need to have $k_p \cdot \Delta < 2\pi$ to avoid phase ambiguity, where $\Delta = +1 - (-1) = 2$
Therefore k_p has to be less than $\frac{2\pi}{\Delta} = \pi$ #

3. $\boxed{5.1\text{-}3}$ $\omega_c = 2\pi \times 10^3$ rad/s



(a) $k_f = 20\pi$

$S_{FM}(t) = A_c \cos\left(\omega_c t + k_f \int_{-\infty}^{t} m(\tau)d\tau + \theta_0\right)$, with $A_c = 1$, $\theta_0 = 0$

$f_i = 10^3 + 10m(t)$, $f_i \in [10^3 - 10\times 3, 10^3 + 10\times 3] = [970, 1030]$ Hz

(b) $k_p = \frac{\pi}{2}$.

$S_{PM}(t) = A_c \cos(\omega_c t + k_p m(t) + \theta_0)$, with $A_c = 1$, $\theta_0 = 0$.

$f_i = 10^3 + \frac{1}{4}\frac{dm(t)}{dt}$, $\frac{dm(t)}{dt} \in [-60, 60] \Rightarrow f_i \in [10^3 - \frac{60}{4}, 10^3 + \frac{60}{4}] = [985, 1015]$ Hz

4. $\boxed{5.1\text{-}4}$ $\varphi_{EM}(t) = 10\cos(13000\pi t)$, $|t| \le 1$

$\omega_c = 10000\pi$

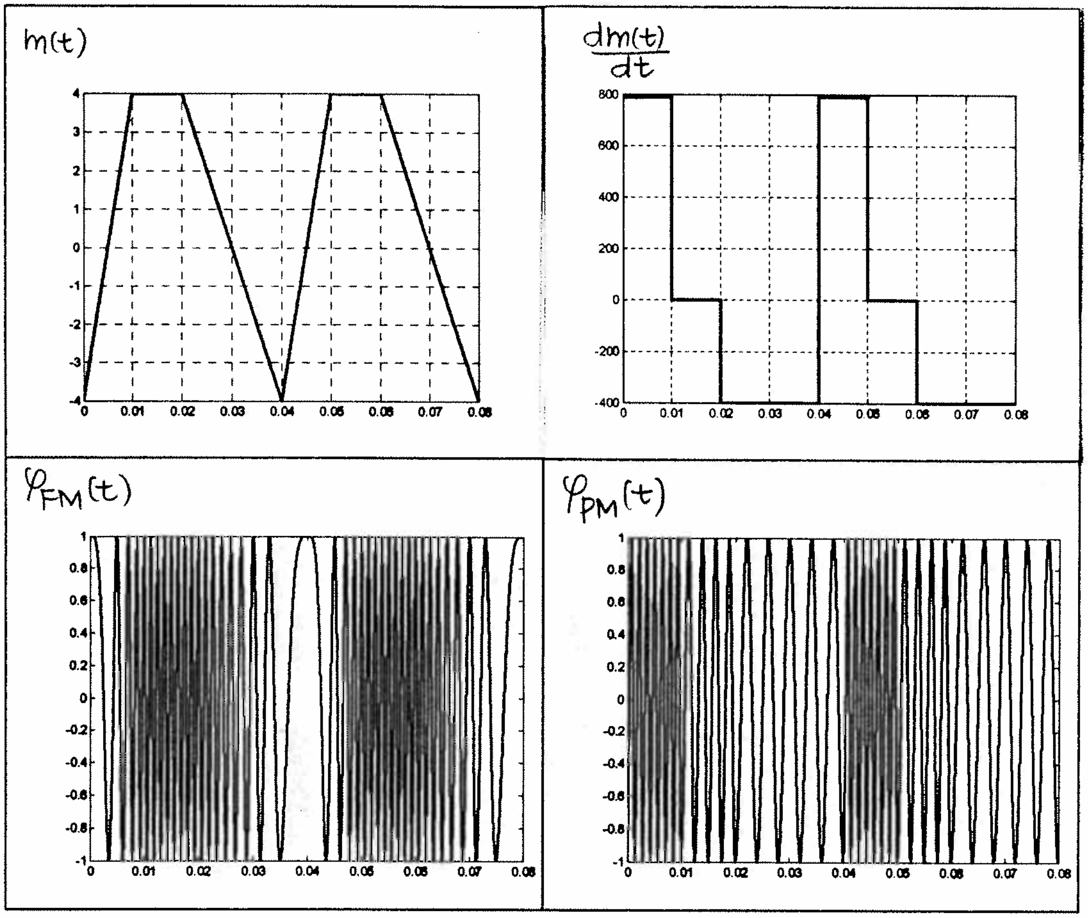
(a) $k_p = 1000$, $\omega_c t + k_p m(t) = 13000\pi t \Rightarrow m(t) = \frac{1}{k_p}[13000\pi t - \omega_c t] = 3\pi t$, $|t| \le 1$

(b) $k_f = 1000$, $\omega_c t + k_f \int_{-\infty}^{t} m(\tau)d\tau = 13000\pi t \Rightarrow \int_{-\infty}^{t} m(\tau)d\tau = \frac{1}{k_f}[13000\pi t - \omega_c t]$

$= 3\pi t$, $|t| \le 1$.

$\Rightarrow m(t) = \frac{d(3\pi t)}{dt} = 3\pi$, $|t| \le 1$ #

5. $\boxed{5.1\text{-}5}$ $\omega_c = 4\pi \times 10^3$ rad/s



(a) $k_f = 500\pi$

$\varphi_{FM}(t) = A_c \cos\left(\omega_c t + k_f \int_{-\infty}^{t} m(\tau)d\tau + \theta_0\right)$, with $A_c = 1$, $\theta_0 = 0$.

$f_i = 2000 + 250\, m(t) \Rightarrow f_i \in [2000 - 250 \times 4,\ 2000 + 250 \times 4] = [1000, 3000]$ Hz

(b) $k_p = 0.25\pi$

$\varphi_{PM}(t) = A_c \cos(\omega_c t + k_p m(t) + \theta_0)$, with $A_c = 1$, $\theta_0 = 0$,

$f_i = 2000 + \frac{1}{8}\frac{dm(t)}{dt}$, Since $\frac{dm(t)}{dt} \in [-400, 800]$, we have

$f_i \in [2000 - \frac{1}{8} \times 400,\ 2000 + \frac{1}{8} \times 800] = [1950, 2100]$ Hz.

6. $\boxed{5.2\text{-}1}$

For Prob. 5.1-3, bandwidth of m(t) is approximated by its own 5th-harmonic frequency. Hence $B = \frac{5}{T} = \frac{5}{0.25} = 20\ Hz$.

(a) For FM signal, $B_{FM} = 2(\Delta f + B)$. $\Delta f = \frac{k_f m_p}{2\pi} = \frac{20\pi \times 3}{2\pi} = 30\ Hz$ (5.13)

$\Rightarrow B_{FM} = 2(30 + 20) = 100\ Hz$

(b) For PM signal, $B_{PM} = 2(\Delta f + B)$, $\Delta f = \frac{k_p \dot{m}_p}{2\pi} = \frac{(\pi/2) \times 60}{2\pi} = 15\ Hz$ (5.18)

$\Rightarrow B_{PM} = 2(15 + 20) = 70\ Hz$

7. $\boxed{5.2\text{-}2}$

For Prob. 5.1-5, assume bandwidth of m(t) is its 5th-harmonic frequency. That is, $B = \frac{5}{T} = \frac{5}{0.04} = 125\ Hz$

(a) For FM signal, $B_{FM} = 2(\Delta f + B)$, $\Delta f = \frac{k_f m_p}{2\pi} = \frac{500\pi \times 4}{2\pi} = 1000\ Hz$ (5.13)

$\Rightarrow B_{FM} = 2(1000 + 125) = 2250\ Hz$

(b) For PM signal, $B_{PM} = 2(\Delta f + B)$, $\Delta f = \frac{k_p[\dot{m}(t)_{max} - \dot{m}(t)_{min}]}{2 \times 2\pi}$ (5.17a)

$$= \frac{(\frac{\pi}{4})[800 - (-400)]}{4\pi} = 75\ Hz$$

$\Rightarrow B_{PM} = 2(75 + 125) = 400\ Hz$

8. $\boxed{5.2\text{-}4}$

$\varphi_{EM}(t) = 10\cos(\omega_c t + 0.1\sin 2000\pi t)$, $\omega_c = 2\pi \times 10^6$

(a) $P = \frac{1}{2} \times 10^2 = 50$ (P. 256 of textbook)

(b) $\theta(t) = \omega_c t + 0.1\sin(2000\pi t) \Rightarrow \omega_i(t) = \frac{d\theta(t)}{dt} = \omega_c + 200\pi\cos(2000\pi t)$

$\Rightarrow \Delta f = \frac{200\pi}{2\pi} = 100\ Hz$ (frequency deviation)

(c) $\theta(t) = \omega_c t + 0.1\sin(2000\pi t)$

$\Rightarrow$ phase deviation $\Delta\phi = 0.1$ rad

(d) $B = \frac{2000\pi}{2\pi} = 1000\ Hz$

$\Rightarrow B_{EM} = 2(\Delta f + B) = 2(100 + 1000) = 2200\ Hz$

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9.
$$J_k(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - kx)}\,dx \quad , \quad k = 0, \pm 1, \pm 2, \cdots, \ \beta \in \mathbb{R}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cos(\beta\sin x - kx) + j\sin(\beta\sin x - kx)\right]dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\beta\sin x - kx)\,dx + j\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(\beta\sin x - kx)\,dx$$

We note $\cos(\beta\sin(-x) - k(-x)) = \cos(-\beta\sin x + kx) = \cos(\beta\sin x - kx)$

$\sin(\beta\sin(-x) - k(-x)) = \sin(-\beta\sin x + kx) = -\sin(\beta\sin x - kx)$

Thus, $\cos(\beta\sin x - kx)$ is an even function of x

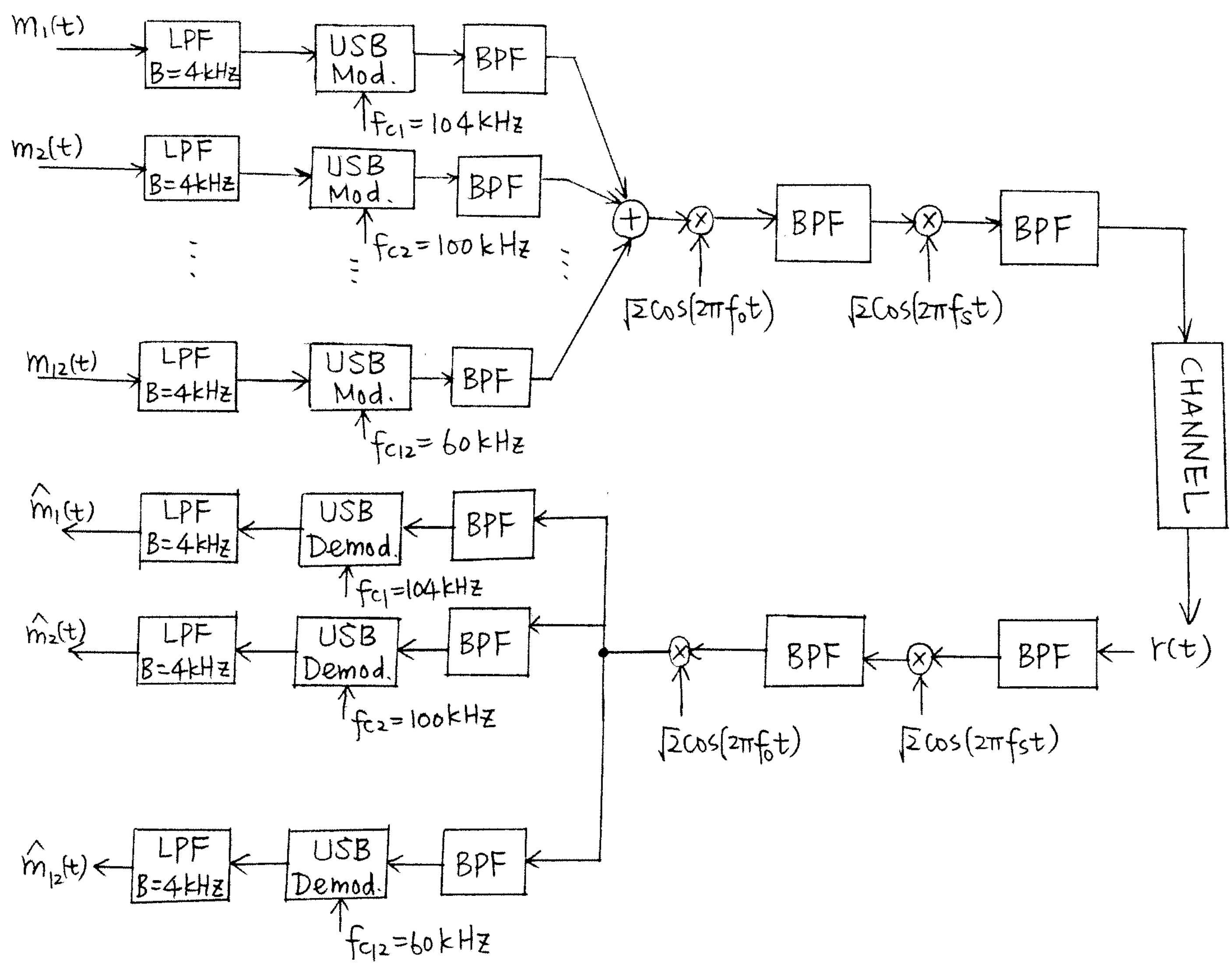
$\sin(\beta\sin x - kx)$ is an odd function of x

$$\Rightarrow J_k(\beta) = \frac{1}{2\pi}\times 2\int_0^{\pi}\cos(\beta\sin x - kx)\,dx + j\times\frac{1}{2\pi}\times 0$$

$$= \frac{1}{\pi}\int_0^{\pi}\cos(\beta\sin x - kx)\,dx \ \in \mathbb{R}$$

Therefore, $J_k(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - kx)}\,dx$ is real, $\forall\ \beta \in \mathbb{R}$.
$k = 0, \pm 1, \cdots$

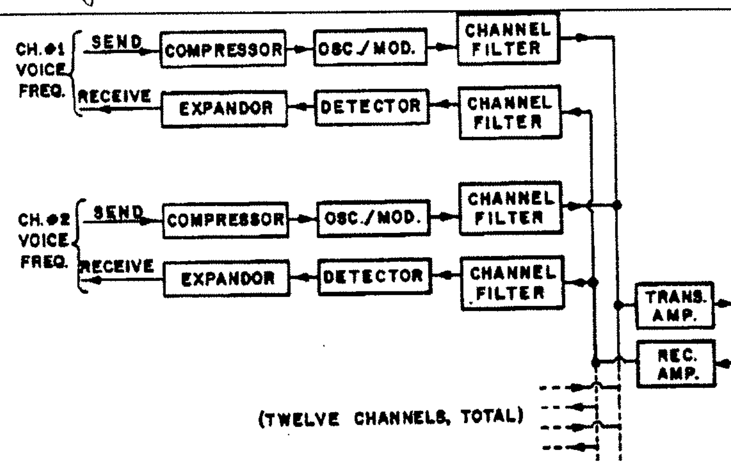
10. Take USB modulation/demodulation for example.



Note: f_0 is determined by Group number and f_s is determined by Super-group number.

10. (Continued)

Also refer to figures shown below [1]:



·Figure 1: Block diagram of DSB FDM transmission system

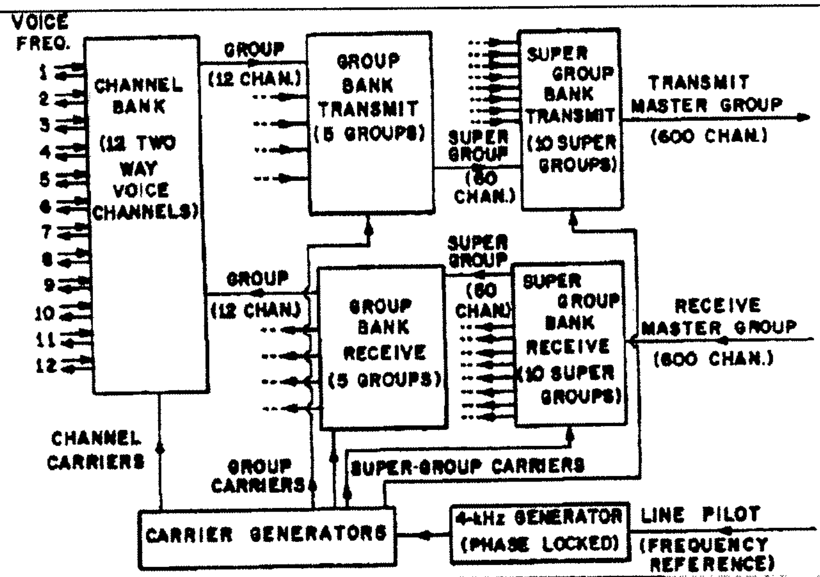


Figure 2(a): Block diagram of SSB-SC FDM long-haul transmission system

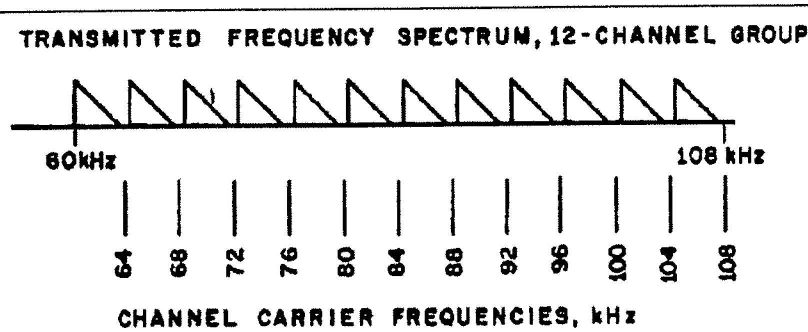


Figure 2(b): The corresponding frequency plan for 12-channel SSB group

REFERENCE

[1] W. L. Smith, "Frequency and Time in Communications," Proc. IEEE, vol. 60, pp. 589-594, May 1972.