HW # 7 Solutions.

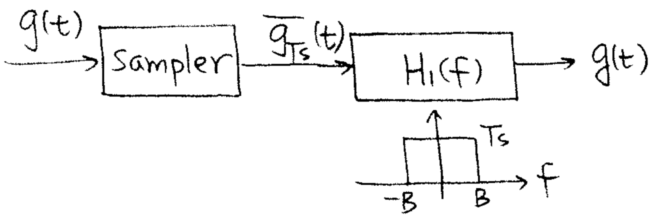
1. **Method 1** (redo derivation) :

Let $\overline{g}_{T_s}(t) = \sum_n g(nT_s)\,\delta(t-nT_s) = g(t)\,\delta_{T_s}(t)$ with $\delta_{T_s}(t) = \sum_n \delta(t-nT_s)$

Since $\delta_{T_s}(t)$ is periodic with period T_s, it admits FS expansion :

$$\delta_{T_s}(t) = \sum_n C_n e^{+j2\pi \frac{n}{T_s} t}, \quad \text{with } C_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_{T_s}(t)\, e^{-j2\pi \frac{n}{T_s} t}\, dt = \frac{1}{T_s}$$

$$\Rightarrow \overline{G}_{T_s}(f) = \mathcal{F}(\overline{g}_{T_s}(t)) = \frac{1}{T_s}\sum_n \int_{\mathbb{R}} g(t)\, e^{-j2\pi (f-\frac{n}{T_s})t}\, dt = \frac{1}{T_s}\sum_n G(f-\tfrac{n}{T_s}), \quad f \in \mathbb{R}$$



Given $\frac{1}{T_s} > 2B$, $h_1(t) = \mathcal{F}^{-1}(H_1(f)) = 2T_sB\,\frac{\sin(2\pi tB)}{2\pi tB}$. We can reconstruct $g(t)$ by calculating $(h_1 * \overline{g}_{T_s})(t)$.

$$\begin{aligned} g(t) = (h_1 * \overline{g}_{T_s})(t) &= \int_{\mathbb{R}} h_1(s)\,\overline{g}_{T_s}(t-s)\, ds \\ &= \int_{\mathbb{R}} h_1(s)\Big(\sum_n g(nT_s)\,\delta(t-s-nT_s)\Big)\, ds \\ &= \sum_n g(nT_s)\, h_1(t-nT_s) \\ &= \sum_n g(nT_s)\, 2T_sB\,\frac{\sin(2\pi(t-nT_s)B)}{2\pi(t-nT_s)B} \end{aligned}$$

Method 2 (observation) :

Let $\frac{1}{T_s} = 2B'$ where $B' > B$.

$g \in LP(B) \Rightarrow g \in LP(B')$

By directly applying result of lecture note, we have

$$g(t) = \sum_n g(nT_s)\, h_2(t-nT_s), \quad \text{where } h_2(t) \triangleq 2T_sB'\,\frac{\sin(2\pi tB')}{2\pi tB'} = \frac{\sin(2\pi tB')}{2\pi tB'}$$

$$= \sum_n g(nT_s)\,\frac{\sin(2\pi B'(t-nT_s))}{2\pi B'(t-nT_s)}, \quad \text{where } B' = \frac{1}{2T_s} > B.$$

2. $\boxed{6.1\text{-}2}$ $\boxed{\begin{array}{l} \text{sinc}(\pi t) \rightleftharpoons \text{rect}(f) \\ \text{sinc}^2(\pi t) \rightleftharpoons \Delta(\frac{f}{2}) \end{array}}$ Let B denote signal cutoff frequency.

(a) $\text{sinc}(2100\pi t) \rightleftharpoons \frac{1}{2100}\text{rect}(\frac{f}{2100}) \Rightarrow B = \frac{2100}{2} = 1050\ \text{Hz} \Rightarrow \begin{cases} f_{Nyq} = 2B = 2100\ \text{Hz} \\ T_{Nyq} = \frac{1}{2100}\ \text{sec} \end{cases}$

(b) $5\,\text{sinc}^2(200\pi t) \rightleftharpoons \frac{5}{200}\Delta(\frac{f}{400}) \Rightarrow B = \frac{400}{2} = 200\ \text{Hz} \Rightarrow \begin{cases} f_{Nyq} = 2B = 400\ \text{Hz} \\ T_{Nyq} = \frac{1}{400}\ \text{Hz} \end{cases}$

(c) $\text{sinc}(2100\pi t) + \text{sinc}^2(200\pi t)$

from (a)(b), $B = \max(1050, 200) = 1050\ \text{Hz} \Rightarrow \begin{cases} f_{Nyq} = 2B = 2100\ \text{Hz} \\ T_{Nyq} = \frac{1}{2100}\ \text{sec} \end{cases}$

(d) $\text{sinc}(200\pi t) \rightleftharpoons \frac{1}{200}\text{rect}(\frac{f}{200})$, corresponding bandwidth $= 200/2 = 100$ Hz

also, $\text{sinc}(2100\pi t)$ has bandwidth 1050 Hz from (a)

Multiplication in time domain means convolution in frequency domain. Bandwidth of the convolved signal equals the sum of bandwidths of two individual signals. Hence we have $B = 100 + 1050 = 1150$ Hz

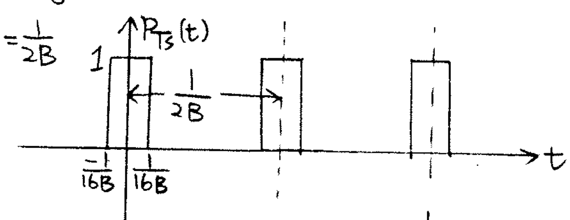
$\Rightarrow \begin{cases} f_{Nyq} = 2B = 2300\ \text{Hz} \\ T_{Nyq} = \frac{1}{2300}\ \text{sec} \end{cases}$

3. $\boxed{6.1\text{-}3}$

We have $\bar{g}(t) = g(t)\sum_{n=-\infty}^{\infty} p(t - nT_s) = g(t)P_{T_s}(t)$

$P_{T_s}(t)$ is periodic with period $T_s = \frac{1}{2B}$

Hence it admits FS expansion



$$P_{T_s}(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi\frac{n}{T_s}t}$$

$$C_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{T_s/2} P_{T_s}(t)e^{-j2\pi\frac{n}{T_s}t}dt = 2B\int_{\frac{-1}{16B}}^{\frac{1}{16B}} e^{-j4\pi nBt}dt = \begin{cases} \frac{1}{4}, & n = 0 \\ \frac{1}{n\pi}\sin(\frac{n\pi}{4}), & n \neq 0 \end{cases}$$

$$\Rightarrow P_{T_s}(t) = \sum_{n=-\infty}^{\infty} C_n e^{j4\pi nBt} = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(\frac{n\pi}{4})\cos(4\pi nBt)$$

$$\Rightarrow \bar{g}(t) = g(t)P_{T_s}(t) = \frac{1}{4}g(t) + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(\frac{n\pi}{4})g(t)\cos(4\pi nBt)$$

$g \in LP(B)$ $$= \frac{1}{4}g(t) + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(\frac{n\pi}{4})g(t)\cos(2\pi(nf_s)t), \quad f_s = \frac{1}{T_s} = 2B$$

Hence $g(t)$ can be recovered by passing $\bar{g}(t)$ through $H(f) = \begin{cases} 4, & |f| \leq B \\ 0, & |f| > B \end{cases}$,

$H(f)$ is an ideal LPF with gain 4 and bandwidth B #

4. $\boxed{6.1\text{-}4}$

$$\boxed{\begin{array}{ll} \text{sinc}(\pi t) \rightleftharpoons \text{rect}(f), & \text{rect}(t) \rightleftharpoons \text{sinc}(\pi f) \\ \text{sinc}^2(\pi t) \rightleftharpoons \Delta\left(\frac{f}{2}\right), & \Delta\left(\frac{t}{2}\right) \rightleftharpoons \text{sinc}^2(\pi f) \end{array}}, \quad f_s = \frac{1}{T_s} > 2B$$

Use results of section 6.1.1, the Equalizer filter $E(f) = \begin{cases} \frac{T_s}{P(f)}, & |f| \le B \\ \text{flexible}, & B < |f| < f_s - B \\ 0, & |f| \ge f_s - B \end{cases}$

(a) $p(t) = \text{rect}\left(\frac{t}{T_s} - \frac{1}{2}\right) = \text{rect}\left(\frac{t - T_s/2}{T_s}\right)$

$$p(t) \rightleftharpoons P(f) = T_s\, \text{sinc}(\pi T_s f)\, e^{-j2\pi f \frac{T_s}{2}} = T_s\, \text{sinc}(\pi f T_s)\, e^{-j\pi f T_s}$$

$$\Rightarrow E(f) = \begin{cases} \text{sinc}^{-1}(\pi f T_s)\, e^{j\pi f T_s}, & |f| \le B \\ \text{Flexible}, & B < |f| < f_s - B \\ 0, & |f| \ge f_s - B \end{cases}$$

(b) $p(t) = \text{rect}\left(\frac{t}{(T_s/2)} - \frac{1}{2}\right) = \text{rect}\left(\frac{t - T_s/4}{(T_s/2)}\right) \rightleftharpoons P(f) = \frac{T_s}{2}\,\text{sinc}\left(\frac{\pi f T_s}{2}\right) e^{-j\pi f T_s/2}$

$$\Rightarrow E(f) = \begin{cases} 2\,\text{sinc}^{-1}\left(\frac{\pi f T_s}{2}\right) e^{j\pi f T_s/2}, & |f| \le B \\ \text{flexible}, & B < |f| < f_s - B \\ 0, & |f| \ge f_s - B \end{cases}$$

(c) $p(t) = \sin\left(2\pi \frac{t}{T_s}\right)\left[u(t) - u(t - T_s/2)\right] = \sin\left(2\pi \frac{t}{T_s}\right) u(t) + \sin\left(2\pi \frac{(t - \frac{T_s}{2})}{T_s}\right) u\left(t - \frac{T_s}{2}\right)$

From Table 3.1, $P(f) = \left[\frac{1}{4j}\left(\delta\left(f - \frac{1}{T_s}\right) - \delta\left(f + \frac{1}{T_s}\right)\right) + \frac{2\pi/T_s}{(2\pi/T_s)^2 - (2\pi f)^2}\right]\left(1 + e^{-j\pi f T_s}\right)$

$$\Rightarrow P(f) = \frac{1}{2\pi} \frac{\left(\frac{1}{T_s}\right)}{\left(\frac{1}{T_s}\right)^2 - f^2}\left(1 + e^{-j\pi f T_s}\right) \quad \text{for } |f| \le B$$

$$\Rightarrow E(f) = \begin{cases} 2\pi(1 - f^2 T_s^2)(1 + e^{-j\pi f T_s})^{-1}, & |f| \le B \\ \text{flexible}, & B < |f| < f_s - B \\ 0, & |f| \ge f_s - B \end{cases}$$

(d) $p(t) = \Delta\left(\frac{t}{T_s/2} - \frac{1}{2}\right) = \Delta\left(\frac{t - T_s/4}{T_s/2}\right)$

$$P(f) = \frac{T_s}{4}\,\text{sinc}^2\left(\frac{T_s \pi f}{4}\right) e^{-j2\pi f (T_s/4)} = \frac{T_s}{4}\,\text{sinc}^2\left(\frac{\pi f T_s}{4}\right) e^{-j\pi f T_s/2}$$

$$\Rightarrow E(f) = \begin{cases} 4\,\text{sinc}^{-2}\left(\frac{\pi f T_s}{4}\right) e^{j\pi f T_s/2}, & |f| \le B \\ \text{flexible}, & B < |f| < f_s - B \\ 0, & |f| \ge f_s - B \end{cases}$$

(e) The recovered signal equals the original signal $g(t)$.

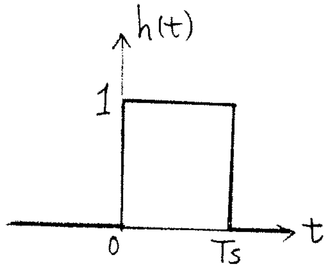
5. $\boxed{6.1\text{-}6}$

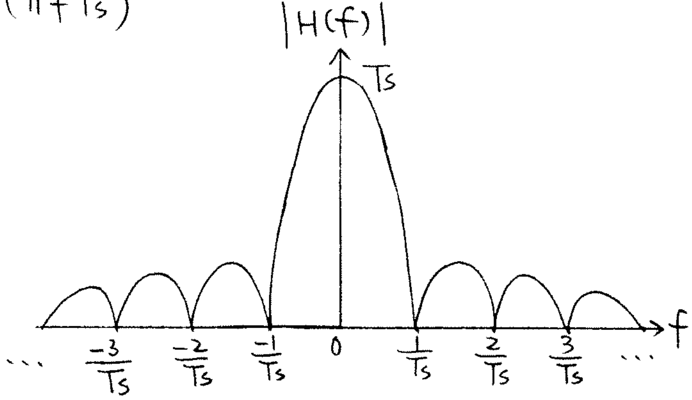
(a) let the input be $\delta(t)$, then output equals $h(t)=\int_{-\infty}^{t}\delta(u)du-\int_{-\infty}^{t-T_s}\delta(u)du$

$$\Rightarrow h(t)=u(t)-u(t-T_s)=\text{rect}\left(\frac{t-T_s/2}{T_s}\right)$$

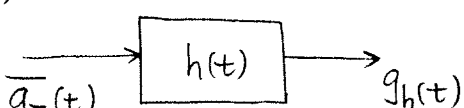
(b) $H(f)=\mathcal{F}(h(t))=T_s\,\text{sinc}(\pi T_s f)e^{-j2\pi f T_s/2}=T_s\,\text{sinc}(\pi f T_s)e^{-j\pi f T_s}$

$|H(f)|=T_s\,\text{sinc}(\pi f T_s)$





(c)



The output $g_h(t)=(h*\bar{g}_{T_s})(t)=\int_{\mathbb{R}}h(t-\tau)\bar{g}_{T_s}(\tau)d\tau=\sum_n g(nT_s)\int_{\mathbb{R}}h(t-\tau)\delta(\tau-nT_s)d\tau$

$$=\sum_n g(nT_s)h(t-nT_s)$$

Therefore the output is a staircase approximation of $g(t)$.

6. $\boxed{6.1\text{-}10}$ (see P.315~318 of textbook)

$h(t)=q(t)=\Delta\left(\frac{t}{T_s/4}\right)\Leftrightarrow H(f)=\frac{T_s}{8}\text{sinc}^2\left(\frac{\pi f T_s}{8}\right)$, averaging filter is $h(t)$.

① Let $r(t)$ be the weighting function satisfying $r(t)=0$ for $t\notin(-0.5T,0.5T)$

Then $r_{T_s}(t)\triangleq\sum_{n=-\infty}^{\infty}r(t-nT_s)$ is periodic with period T_s, and it admits FS:

$$r_{T_s}(t)=\sum_{n=-\infty}^{\infty}Q_n e^{j2\pi nt/T_s},\quad \text{where } Q_n=\frac{1}{T_s}\int_{-0.5T}^{0.5T}r(t)e^{-j\left(\frac{2\pi nt}{T_s}\right)}dt$$

② Let $g_1(t)$ be output signal of the averaging filter, $g_1(t)=h(t)*\left(g(t)r_{T_s}(t)\right)$

$$\Rightarrow G_1(f)=H(f)\sum_{n=-\infty}^{\infty}Q_n G(f-nf_s),\quad f_s=\frac{1}{T_s}$$

$$=\frac{T_s}{8}\text{sinc}^2\left(\frac{\pi f T_s}{8}\right)\sum_{n=-\infty}^{\infty}Q_n G(f-nf_s)$$

③ Next, assume we use a practical reconstruction pulse $p(t)$. That is, let us reconstruct $g(t)$ by $\tilde{g}(t)=\sum_m g_1(mT_s)p(t-mT_s)$.

We have $\tilde{G}(f)=\frac{P(f)}{T_s}\sum_{m=-\infty}^{\infty}G_1(f+mf_s)$ (next page)

6. $\boxed{6.1\text{-}10}$ (continued)

$$\Rightarrow \tilde{G}(f) = P(f)\cdot\frac{1}{T_s}\sum_{m=-\infty}^{\infty}\frac{T_s}{8}\,\mathrm{sinc}^2\!\left(\frac{\pi(f+mf_s)T_s}{8}\right)\sum_{n=-\infty}^{\infty}Q_n\,G(f+(m-n)f_s)$$

$$= P(f)\left[\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\frac{1}{8}\,\mathrm{sinc}^2\!\left(\frac{\pi(f+mf_s)T_s}{8}\right)Q_n\,G(f+(m-n)f_s)\right]$$

$\ell = m-n$

$$= P(f)\left[\sum_{n=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\frac{1}{8}\,\mathrm{sinc}^2\!\left(\frac{\pi(f+(\ell+n)f_s)T_s}{8}\right)Q_n\,G(f+\ell f_s)\right]$$

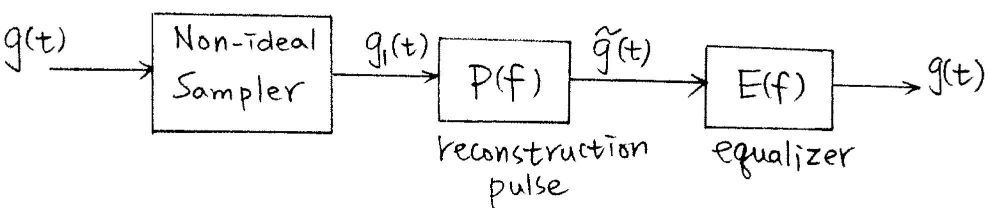
$$= P(f)\left[\sum_{\ell=-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty}\frac{1}{8}\,\mathrm{sinc}^2\!\left(\frac{\pi(f+(\ell+n)f_s)T_s}{8}\right)Q_n\right)G(f+\ell f_s)\right]$$

$$= P(f)\left[\sum_{\ell=-\infty}^{\infty}F_\ell(f)\,G(f+\ell f_s)\right],\ \text{where } F_\ell(f)\triangleq\frac{1}{8}\sum_{n=-\infty}^{\infty}Q_n\,\mathrm{sinc}^2\!\left(\frac{\pi(f+(\ell+n)f_s)T_s}{8}\right)$$

④ Assume $G(f)$ has bandwidth B Hz, we will need to design an equalizer with transfer function $E(f)$ satisfying the following

$$E(f)P(f)F_0(f) = \begin{cases} 1, & |f|<B \\ \text{Flexible}, & B<|f|<f_s-B \\ 0, & |f|>f_s-B \end{cases} \qquad (6.28)$$

where $F_0(f)=\frac{1}{8}\sum_{n=-\infty}^{\infty}Q_n\,\mathrm{sinc}^2\!\left(\frac{\pi(f+nf_s)T_s}{8}\right)$. The block diagram is shown below.



7. $\boxed{6.2\text{-}1}$ (We use $10\log_{10}x$ to represent x in dB)

(a) $L = 65536 = 2^{16}$. Hence, 16 binary digits are required to encode a sample.

(b) (6.34) $SQNR = 3L^2 \frac{\widetilde{m^2(t)}}{m_p^2} = 3 \times (65536)^2 \times \frac{(0.1)}{1^2} = 91.1008$ dB

(c) The Nyquist rate $= 15k \times 2 = 30k$ Hz $\Rightarrow$ bit rate $= 30k \times 16 = 480000$ bits/sec

(d) If $f_s = 44100$ Hz, then bit rate $= 44100 \times 16 = 705600$ bits/sec, and the minimal bandwidth $= \frac{705600}{2} = 352800$ Hz is required to transmit the encoded signal

8. $\boxed{6.2\text{-}2}$

(a) Nyquist rate $= 4.5M \times 2 = 9$ MHz $\Rightarrow f_s = 9$ MHz $\times 1.2 = 10.8$ MHz

(b) $L = 1024 = 2^{10}$. Hence, 10 binary pulses are required to encode a sample.

(c) Bit rate $= 10.8M \times 10 = 108$ M bits/sec, and the mininal bandwidth required to transmit this signal is $\frac{108M}{2} = 54$ MHz.

9. $\boxed{6.2\text{-}3}$

(a) (6.31) The maximum quantization error $= \frac{\Delta V}{2}$, where $\Delta V = \frac{2m_p}{L}$

$$= \frac{m_p}{L} \underset{\uparrow \text{ requirement}}{\leq} 0.25\% \, m_p = \frac{m_p}{400}$$

$\Rightarrow L \geq 400$, where L has to be a power of 2.

Because $400 \in [256, 512] = [2^8, 2^9]$, we choose $L_{min} = 512 = 2^9$; in other words, 9 is the minimum number of bits needed for a uniform quantizer.

(b) Minimum bit rate of the multiplexed data is $\underbrace{15000 \times 2 \times 2}_{\text{Nyquist rate for two (left \& right) signals}} \times 1.2 \times 9 \times 128$

$= 82944000$ bits/sec

(c) With 5% more bits added, the minimum bandwidth to transmit the final data stream is $(82944000 \times 1.05)/2 = 43545600$ Hz

10. $\boxed{6.2\text{-}4}$ (We use $10\log_{10}x$ to represent x in dB)

From (6.34), $SQNR = 3L^2 \dfrac{\widetilde{m^2(t)}}{m_p} = 3L^2 \dfrac{20\times10^{-3}}{1} \underset{\text{requirement}}{\geq} 10^{43/10} = 10^{4.3}$

$\Rightarrow L \geq \sqrt{10^{4.3}/(3\times0.02)} \cong 576.667 \in [512, 1024] = [2^9, 2^{10}]$

$\Rightarrow L_{min} = 2^{10}$, i.e., the minimum number of bits needed is 10;

and $SQNR\left(\text{for } L_{min}\right) = 3\times(2^{10})^2\times\dfrac{20\times10^{-3}}{1} = 47.9875$ dB.

$\boxed{6.2\text{-}6}$

$\begin{cases} \text{The power of } m(t) \text{ in Fig. P.2.8-4(b)} = \frac{1}{2\pi}\times2\times\int_0^{\pi} e^{-\frac{t}{5}}dt = \frac{5}{\pi}\left[1-e^{-\frac{\pi}{5}}\right] \\ m_p = 1 \end{cases}$

$SQNR = 3L^2\cdot\frac{5}{\pi}\left[1-e^{-\frac{\pi}{5}}\right] \overset{\text{requirement}}{\geq} 10^{4.3} \Rightarrow L \geq \sqrt{10^{4.3}/\left(3\times\frac{5}{\pi}\times\left(1-e^{-\frac{\pi}{5}}\right)\right)} \cong 94.645$

$94.645 \in [64, 128] = [2^6, 2^7]$

$\Rightarrow L_{min} = 2^7$, i.e., the minimum number of bits needed is 7;

and $SQNR\left(\text{for } L_{min}\right) = 3\times(2^7)^2\times\frac{5}{\pi}\left[1-e^{-\frac{\pi}{5}}\right] = 45.6222$ dB.

$\boxed{6.2\text{-}7}$ $\mu^2 = 10^4 \gg \dfrac{1^2}{0.02} = 50$.

From (6.36) SQNR of μ-law compandor is $\dfrac{3L^2}{[\ln(1+\mu)]^2} \underset{\text{requirement}}{\geq} 10^{4.3}$, where $\mu = 100$.

$\Rightarrow L \geq \sqrt{10^{4.3}\times\left(\ln(1+100)\right)^2/3} \cong 376.376$

$376.376 \in [256, 512] = [2^8, 2^9]$

$\Rightarrow L_{min} = 2^9$; i.e., the minimum number of bits needed is 9;

and $SQNR\left(\text{for } L_{min}\right) = \dfrac{3\times(2^9)^2}{[\ln(1+100)]^2} = 45.673$ dB

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