ENEE 420
FALL 2007
COMMUNICATIONS SYSTEMS

SAMPLING

In these notes we discuss the sampling process and properties of some of its mathematical description. This culminates in the celebrated Shannon-Nyquist Sampling Theorem.

Rectangular pulses

With $\tau > 0$, the rectangular pulse $p_\tau : \mathbb{R} \to \mathbb{R}$ is defined by

$$p_\tau(t) = \begin{cases} 1 & \text{if } 0 \le t \le \tau \\ 0 & \text{otherwise.} \end{cases}$$

Fourier analysis of rectangular pulses

For each $f \neq 0$ in $\mathbb{R}$, straightforward calculations show

$$\begin{aligned} P_\tau(f) &= \int_{\mathbb{R}} p_\tau(t) e^{-j2\pi f t} dt \\ &= \int_0^\tau e^{-j2\pi f t} dt \\ &= \frac{e^{-j2\pi f \tau} - 1}{-j2\pi f} \\ &= \frac{\sin(\pi f \tau)}{\pi f} \cdot e^{-j\pi f \tau} \end{aligned} \tag{1}$$

while

$$P_\tau(f) = \tau, \quad f = 0.$$

Therefore,

$$P_\tau(f) = \begin{cases} \frac{\sin(\pi f \tau)}{\pi f} \cdot e^{-j\pi f \tau} & \text{if } f \neq 0 \\ \tau & \text{if } f = 0. \end{cases} \tag{2}$$

From now on, the parameters τ and T_s are selected so that $0 < \tau < T_s$, and we write $f_s = \frac{1}{T_s}$.

Throughout, we use the notation $\sum_k$ to denote the summation $\sum_{k=0,\pm 1,\ldots}$ over all integers.

Train of rectangular pulses

The train of pulses associated with p_τ is the signal $c_\tau : \mathbb{R} \to \mathbb{R}$ given by

$$c_\tau(t) := \sum_k p_\tau(t - kT_s), \quad t \in \mathbb{R}.$$

The signal c_τ being periodic with period T_s, it admits a Fourier series representation, namely

$$c_\tau(t) = \sum_k \alpha_\tau(k) e^{j2\pi k f_s t}, \quad t \in \mathbb{R}$$

with Fourier coefficients given by

$$\begin{aligned} \alpha_\tau(k) &= \frac{1}{T_s} \int_0^{T_s} p_\tau(t) e^{-j2\pi k f_s t} dt \\ &= \frac{1}{T_s} P_\tau(k f_s), \quad k = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

By virtue of (2) we find

$$\alpha_\tau(k) = \begin{cases} \frac{1}{T_s} \frac{\sin(\pi k f_s \tau)}{\pi k f_s} \cdot e^{-j\pi k f_s \tau} & \text{if } k = \pm 1, \pm 2, \ldots \\ \frac{\tau}{T_s} & \text{if } k = 0. \end{cases} \tag{3}$$

It is now plain that

$$\frac{c_\tau(t)}{\tau} = \sum_k \frac{\alpha_\tau(k)}{\tau} \cdot e^{j2\pi k f_s t}, \quad t \in \mathbb{R}. \tag{4}$$

Natural sampling

Let $g : \mathbb{R} \to \mathbb{R}$ denote an information-bearing signal. *Natural sampling* gives rise to the signal $g_{\text{Nat},\tau} : \mathbb{R} \to \mathbb{R}$ defined by

$$g_{\text{Nat},\tau}(t) = c_\tau(t) g(t) = \sum_k p_\tau(t - kT_s) \cdot g(t), \quad t \in \mathbb{R}.$$

Its Fourier transform is given by

$$
\begin{aligned}
G_{\mathrm{Nat},\tau}(f) &= \int_{\mathbb{R}} g_{\mathrm{Nat},\tau}(t) e^{-j2\pi ft} dt \\
&= \int_{\mathbb{R}} c_\tau(t) g(t) e^{-j2\pi ft} dt \\
&= \int_{\mathbb{R}} \left(\sum_k \alpha_\tau(k) e^{j2\pi k f_s t} \right) g(t) e^{-j2\pi ft} dt \\
&= \sum_k \alpha_\tau(k) \int_{\mathbb{R}} g(t) e^{j2\pi k f_s t} e^{-j2\pi ft} dt \\
&= \sum_k \alpha_\tau(k) \int_{\mathbb{R}} g(t) e^{-j2\pi (f - k f_s) t} dt \\
&= \sum_k \alpha_\tau(k) G(f - k f_s), \quad f \in \mathbb{R}. \qquad (5)
\end{aligned}
$$

As a result, we also find

$$
\begin{aligned}
\frac{G_{\mathrm{Nat},\tau}(f)}{\tau} &= \int_{\mathbb{R}} \frac{g_{\mathrm{Nat},\tau}(t)}{\tau} \cdot e^{-j2\pi ft} dt \\
&= \sum_k \frac{\alpha_\tau(k)}{\tau} \cdot G(f - k f_s), \quad f \in \mathbb{R}. \qquad (6)
\end{aligned}
$$

Ideal pulses

It may seem natural to model an instantaneous (or ideal) pulse (at time $t = 0$) as a mapping (or function) $p : \mathbb{R} \to \mathbb{R}$ such that

$$
p(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \\ 1 & \text{if } t = 0. \end{cases}
$$

Unfortunately, such a definition is not a useful one due to the following fact: From the point of view of Fourier analysis, the function p is indistinguishable from the identically zero function.

Instead, we model an ideal pulse by a Dirac function $\delta : \mathbb{R} \to \mathbb{R}$. We draw attention to the fact that although we present the pulse δ as if it *were* a function $\mathbb{R} \to \mathbb{R}$, this is far from being the case! The terminology is an accepted one and we shall use it throughout.

Formally, we can compute the Fourier transform of the Dirac function as

$$\int_{\mathbb{R}} \delta(t) e^{-j2\pi ft} dt = 1, \quad f \in \mathbb{R}. \tag{7}$$

Train of ideal pulses

In analogy with the notion of train of natural pulses, we can associate with ideal pulses the corresponding notion of pulse train. We define such a train of ideal pulses as the mapping $c_\delta : \mathbb{R} \to \mathbb{R}$ given by

$$c_\delta(t) = \sum_k \delta(t - kT_s), \quad t \in \mathbb{R}.$$

Again caution is in order: While the train c_δ of ideal pulses may have been presented as if it were a mapping $\mathbb{R} \to \mathbb{R}$, this is not so due to the (unresolved) conceptional difficulties mentionned earlier. Yet, despite the fact that such a train of ideal pulses has only been vaguely defined (if at all), this notion does serve a useful purpose, albeit a formal one, as will become apparent below.

Again proceeding *formally*, we compute the Fourier transform of c_δ as

$$\begin{aligned} C_\delta(f) &= \int_{\mathbb{R}} c_\delta(t) e^{-j2\pi ft} dt \\ &= \int_{\mathbb{R}} \left(\sum_k \delta(t - kT_s) \right) e^{-j2\pi ft} dt \\ &= \sum_k \int_{\mathbb{R}} \delta(t - kT_s) e^{-j2\pi ft} dt \\ &= \sum_k e^{-j2\pi fkT_s}, \quad f \in \mathbb{R}. \end{aligned} \tag{8}$$

Ideal sampling

Let $g : \mathbb{R} \to \mathbb{R}$ denote an information-bearing signal. Ideal sampling produces the signal $g_{\text{Ideal}} : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} g_{\text{Ideal}}(t) &= c_\delta(t) g(t) \\ &= \sum_k \delta(t - kT_s) g(t) \\ &= \sum_k g(kT_s) \delta(t - kT_s), \quad t \in \mathbb{R}. \end{aligned} \tag{9}$$

Its Fourier transform is therefore given by

$$\begin{aligned} G_{\text{Ideal}}(f) &= \int_{\mathbb{R}} g_{\text{Ideal}}(t) e^{-j2\pi f t} dt \\ &= \sum_k g(kT_s) \int_{\mathbb{R}} \delta(t - kT_s) e^{-j2\pi f t} dt \\ &= \sum_k g(kT_s) e^{-j2\pi f k T_s} \quad f \in \mathbb{R}. \end{aligned} \tag{10}$$

This expression turns out to be not too useful for our purposes, a state of affairs which prompts us to seek a different approach for evaluating the Fourier transform G_{Ideal}. Although the expressions to be given are formal expressions for the Fourier transform of an object which has not been fully defined, they will turn out to be useful for understanding the properties of the sampling process.

From natural to ideal pulses

For each $\tau > 0$, the normalized rectangular pulse $p^\star_\tau : \mathbb{R} \to \mathbb{R}$ is defined by

$$p^\star_\tau(t) := \frac{p_\tau(t)}{\tau} = \begin{cases} \frac{1}{\tau} & \text{if } 0 \le t \le \tau \\ 0 & \text{otherwise.} \end{cases}$$

Its Fourier transform is simply given by

$$P^\star_\tau(f) := \frac{P_\tau(f)}{\tau} = \begin{cases} \frac{\sin(\pi f \tau)}{\pi f \tau} \cdot e^{-j\pi f \tau} & \text{if } f \neq 0 \\ 1 & \text{if } f = 0. \end{cases}$$

The convergence

$$\lim_{\tau \downarrow 0} P^\star_\tau(f) = \lim_{\tau \downarrow 0} \frac{P_\tau(f)}{\tau} = 1, \quad f \in \mathbb{R}$$

and the expression (7) for the Fourier transform of a Dirac function together suggest that the ideal pulse δ can be thought of as the limit of the normalized pulse $p^\star_\tau$ as τ goes to zero. Conversely, the normalized pulse $p^\star_\tau$ with small τ can be interpreted as a good approximation for the ideal pulse δ. We symbolically summarize such a convergence as

$$\lim_{\tau \downarrow 0} p^\star_\tau = \delta. \tag{11}$$

We make no attempt at giving a precise meaning to the convergence (11). In fact, a precise definition is certainly fraught with difficulties, some of which are already apparent from the pointwise convergence

$$\lim_{\tau\downarrow 0} p^\star_\tau(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ & \\ \infty & \text{if } t = 0. \end{cases}$$

Resolving these difficulties is beyond the scope of these notes.

From natural to ideal sampling

Let τ go to zero: Since

$$\lim_{\tau\downarrow 0} \frac{\sin(\pi k f_s \tau)}{\pi k f_s \tau} = 1, \quad k \neq 0$$

it is plain that

$$\lim_{\tau\downarrow 0} P^\star_\tau(k f_s) = \lim_{\tau\downarrow 0} \frac{\alpha_\tau(k)}{\tau} = 1, \quad k = 0, \pm 1, \pm 2, \ldots .$$

Therefore, formally we conclude that

$$\begin{aligned} \lim_{\tau\downarrow 0} \frac{c_\tau(t)}{\tau} &= \lim_{\tau\downarrow 0} \sum_k \frac{\alpha_\tau(k)}{\tau} e^{j2\pi k f_s t} \\ &= \sum_k \left(\lim_{\tau\downarrow 0} \frac{\alpha_\tau(k)}{\tau} \right) \cdot e^{j2\pi k f_s t} \\ &= \frac{1}{T_s} \sum_k e^{j2\pi k f_s t}. \end{aligned} \tag{12}$$

On the other hand, formally wielding (11) we find

$$\begin{aligned} \lim_{\tau\downarrow 0} \frac{c_\tau(t)}{\tau} &= \lim_{\tau\downarrow 0} \sum_k p^\star_\tau(t - kT_s) \\ &= \sum_k \lim_{\tau\downarrow 0} p^\star_\tau(t - kT_s) \\ &= \sum_k \delta(t - kT_s) \\ &= c_\delta(t), \quad t \in \mathbb{R}. \end{aligned} \tag{13}$$

Combining we conclude that

$$\sum_k \delta(t-kT_s) = \frac{1}{T_s}\sum_k e^{j2\pi k f_s t}, \quad t \in \mathbb{R}.$$

This (formal) relation often appears in the literature, and is known as *Poisson's summation formula*.

From train of natural pulses to train of ideal pulses

Next, we see that

$$\begin{aligned} G_{\rm Ideal}(f) &= \lim_{\tau\downarrow 0}\frac{G_{{\rm Nat},\tau}(f)}{\tau} \\ &= \lim_{\tau\downarrow 0}\sum_k \frac{\alpha_\tau(k)}{\tau}\cdot G(f-kf_s) \\ &= \sum_k \left(\lim_{\tau\downarrow 0}\frac{\alpha_\tau(k)}{\tau}\right)\cdot G(f-kf_s), \quad f\in\mathbb{R}. \end{aligned} \tag{14}$$

In short,

$$G_{\rm Ideal}(f) = \frac{1}{T_s}\sum_k G(f-kf_s), \quad f\in\mathbb{R}. \tag{15}$$

Recovering g from $g_{{\rm Nat},\tau}$

Assume the signal $g:\mathbb{R}\to\mathbb{R}$ to be band-limited with cut-off frequency W, i.e.,

$$G(f) = 0, \quad |f| > W. \tag{16}$$

The frequency

$$f_{\rm Nyq} = 2W$$

plays a particular role and is known as the *Nyquist* rate for g.

Under the condition (16) the translates $G(f-kf_s)$ and $G(f-\ell f_s)$ do not "overlap" if $k\neq\ell$ whenever the condition

$$2W < f_s \tag{17}$$

holds. More precisely, under (16) and (17), the translates $G(f-kf_s)$ and $G(f-\ell f_s)$ with $k\neq\ell$ cannot be simultaneously non-zero. As a result, in the expression for the Fourier transform of $g_{{\rm Nat},\tau}$, namely

$$G_{{\rm Nat},\tau}(f) = \sum_k \alpha_\tau(k)\cdot G(f-kf_s), \quad f\in\mathbb{R},$$

at most one of the terms $G(f - kf_s)$, $k = 0, \pm 1, \pm 2, \ldots$, is ever non-zero for a given frequency f under the condition (17).

With this in mind, consider a lowpass filter H with cutoff frequency W_h, i.e.,

$$H(f) = 0, \quad |f| > W_h.$$

If we select W_h so that

(18) $$W < W_h < f_s - W,$$

then

(19) $$\begin{aligned} H(f) \cdot G_{\mathrm{Nat},\tau}(f) &= \sum\nolimits_k \alpha_\tau(k) \cdot H(f) G(f - kf_s) \\ &= \alpha_\tau(0) H(f) G(f), \quad f \in \mathbb{R} \end{aligned}$$

since for all $k = \pm 1, \ldots$, we have

$$H(f) G(f - kf_s) = 0, \quad f \in \mathbb{R}.$$

In particular, if we take the lowpass filter H to be

$$H(f) = \begin{cases} 1 & \text{if } |f| \leq W_h \\ \\ 0 & \text{otherwise,} \end{cases}$$

then we obtain

$$H(f) \cdot G_{\mathrm{Nat},\tau}(f) = \frac{\tau}{T_s} G(f), \quad f \in \mathbb{R}.$$

Thus, the lowpass information-bearing signal $m : \mathbb{R} \to \mathbb{R}$ can be recovered fully from $g_{\mathrm{Nat},\tau}$ by linear processing.

The Shannon-Nyquist Sampling Theorem

We now show that not only can g also be recovered from g_{Ideal}, but that this signal can be reconstructed from the samples $\{g(kT_s),\ k = 0, \pm 1, \ldots\}$.

Here as well we assume the signal $g : \mathbb{R} \to \mathbb{R}$ to be band-limited with cut-off frequency W. Moreover, the condition (17) is enforced. From (15) we readily conclude that

$$G_{\mathrm{Ideal}}(f) = \frac{1}{T_s} G(f), \quad |f| \leq W$$

so that

$$G(f) = T_s G_{\mathrm{Ideal}}(f), \quad |f| \leq W$$

Reporting this fact into (10) we conclude that

$$G(f) = T_s \sum_k g(kT_s) e^{-j2\pi f k T_s}, \quad |f| \le W$$

Since the signal g is band-limited with cut-off frequency W, this last relation already shows that the *samples* should be sufficient to reconstruct the original signal g! By Fourier inversion, we get

$$\begin{aligned} g(t) &= \int_{\mathbb{R}} G(f) e^{j2\pi f t} dt \\ &= \int_{-W}^{W} G(f) e^{j2\pi f t} dt \\ &= \int_{-W}^{W} \left(T_s \sum_k g(kT_s) e^{-j2\pi f k T_s} \right) e^{j2\pi f t} dt \\ &= T_s \sum_k g(kT_s) \int_{-W}^{W} e^{j2\pi f (t - kT_s)} dt \\ &= T_s \sum_k g(kT_s) \cdot \frac{e^{j2\pi(t-kT_s)W} - e^{-j2\pi(t-kT_s)W}}{j2\pi(t-kT_s)} \end{aligned} \tag{20}$$

so that

$$g(t) = T_s \sum_k g(kT_s) \cdot \frac{\sin\left(2\pi(t-kT_s)W\right)}{\pi(t-kT_s)}, \quad t \in \mathbb{R}. \tag{21}$$

This expression is sometimes written in terms of the Nyquist rate for the signal g, namely

$$\begin{aligned} g(t) &= T_s \sum_k g(kT_s) \cdot \frac{\sin\left(\pi(t-kT_s) f_{\mathrm{Nyq}}\right)}{\pi(t-kT_s)} \\ &= T_s f_{\mathrm{Nyq}} \cdot \sum_k g(kT_s) \cdot \mathrm{sinc}\left((t-kT_s) f_{\mathrm{Nyq}}\right), \quad t \in \mathbb{R} \end{aligned} \tag{22}$$

where we have defined

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}, \quad t \in \mathbb{R}.$$

With $f_s = f_{\mathrm{Nyq}}$, we get $T_s f_{\mathrm{Nyq}} = 1$ and the last relation becomes

$$\begin{aligned} g(t) &= \sum_k g(kT_s) \cdot \mathrm{sinc}\left((t-kT_s) f_{\mathrm{Nyq}}\right) \\ &= \sum_k g(kT_s) \cdot \mathrm{sinc}\left(f_{\mathrm{Nyq}} t - k\right), \quad t \in \mathbb{R}. \end{aligned} \tag{23}$$