ENEE 420
FALL 2010
COMMUNICATIONS SYSTEMS

ANSWER KEY TO TEST # 1:

1.

With scalar $a > 0$, consider the signal $g_a : \mathbb{R} \to \mathbb{R}$ given by

$$g_a(t) := \cos(2\pi t) + \sin(2\pi a t), \quad t \in \mathbb{R}.$$

1.a. For each $T > 0$ we have

$$\begin{aligned} & \int_{-T}^{T} |g_a(t)|^2 dt \\ = & \int_{-T}^{T} |\cos(2\pi t) + \sin(2\pi a t)|^2 \, dt \\ = & \int_{-T}^{T} \left(|\cos(2\pi t)|^2 + 2\cos(2\pi t)\sin(2\pi a t) + |\sin(2\pi a t)|^2 \right) dt. \end{aligned} \tag{1.1}$$

With the help of standard trigonometric identities, elementary calculations yield

$$\begin{aligned} \int_{-T}^{T} |\cos(2\pi t)|^2 \, dt &= \frac{1}{2}\int_{-T}^{T} (1 + \cos(4\pi t)) \, dt \\ &= T + \frac{\sin(4\pi T)}{4\pi} \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \int_{-T}^{T} |\sin(2\pi a t)|^2 \, dt &= \frac{1}{2}\int_{-T}^{T} (1 - \cos(4\pi a t)) \, dt \\ &= T - \frac{\sin(4\pi a T)}{4\pi a}, \end{aligned} \tag{1.3}$$

while

$$\int_{-T}^{T} \cos(2\pi t)\sin(2\pi a t) \, dt = 0$$

since the integrand has odd symmetry with respect to the origin. As a result we conclude that

$$\mathcal{P}_{g_a} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |g_a(t)|^2 dt = 1.$$

1.b. It is assumed that the signal $g_a : \mathbb{R} \to \mathbb{R}$ gives rise to a Fourier series expansion of the form

$$\sum_{n=-\infty}^{\infty} c_n e^{j2\pi nat}, \quad t \in \mathbb{R} \tag{1.4}$$

with Fourier coefficients $\{c_n,\ n = 0, \pm 1, \ldots\}$. The signal g_a being defined on $\mathbb{R}$, the existence of its Fourier series (**??**) implies that g_a must be periodic with period

$$T := \frac{1}{a}.$$

But the signal $t \to \sin(2\pi at)$ being itself periodic with period T, it follows that the signal $t \to \cos(2\pi t)$ must also be periodic with period T. However, the signal $t \to \cos(2\pi t)$ is itself periodic with period 1. These two requirements imply that $T = \ell$ for some integer $\ell = 1, 2, \ldots$, or equivalently

$$a = \frac{1}{\ell}.$$

1.c. Under the condition $a = \frac{1}{\ell}$ for some $\ell = 1, \ldots$, we get the following: If $\ell \neq 1$, then

$$\begin{aligned} g_a(t) &= \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} + \frac{e^{j2\pi at} - e^{-j2\pi at}}{2j} \\ &= \frac{e^{j2\pi \ell at} + e^{-j2\pi \ell at}}{2} + \frac{e^{j2\pi at} - e^{-j2\pi at}}{2j}, \quad t \in \mathbb{R} \end{aligned}$$

so that

$$c_1 = \frac{1}{2j},\ c_{-1} = -\frac{1}{2j} \quad \text{and} \quad c_\ell = c_{-\ell} = \frac{1}{2}$$

with all other Fourier coefficients being zero. If $\ell = 1$, then $a = 1$ and we get

$$c_{\pm 1} = \frac{1}{2}\left(1 \pm \frac{1}{j}\right)$$

with all other Fourier coefficients being zero. In either case we now conclude that

$$\begin{aligned} \mathcal{P}_{g_a} &= \frac{1}{T}\int_0^{\frac{1}{a}} |g_a(t)|^2 dt \quad \text{[By periodicity]} \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \quad \text{[By Parseval's Theorem for Fourier series]} \\ &= \begin{cases} \frac{1}{4}\left|1 - \frac{1}{j}\right|^2 + \frac{1}{4}\left|1 + \frac{1}{j}\right|^2 & \text{if } \ell = 1 \\ \\ 2\left|\frac{1}{2j}\right|^2 + 2\left|\frac{1}{2}\right|^2 & \text{if } \ell \neq 1 \end{cases} \\ &= 1, \end{aligned} \tag{1.5}$$

in agreement with the evaluation carried out in Part **1.a**.

2.

For each $a > 0$, consider the signal $h_a : \mathbb{R} \to \mathbb{R}$ given by

$$h_a(t) = e^{-a|t|}, \quad t \in \mathbb{R}.$$

2.a. By now you should know that

$$H_a(f) = \frac{2a}{a^2 + (2\pi f)^2}, \quad f \in \mathbb{R}$$

and I look forward to seeing your calculations.

2.b. Note that $\lim_{a \downarrow 0} h_a(t) = 1$ for each t in $\mathbb{R}$ and it is easily verified that

$$\lim_{a \downarrow 0} H_a(f) = \begin{cases} 0 & \text{if } f \neq 0 \\ \infty & \text{if } f = 0 \end{cases}$$

so that $\lim_{a \downarrow 0} H_a(f)$ can be viewed as a proxy for $\delta(f)$. Thus, we can construct a direct approximation argument on the way to establish the Fourier pairing $1 \iff \delta(f)$, namely

$$\begin{array}{lcl} h_a(t) & \iff & H_a(f) \\ \downarrow (a \downarrow 0) & & \downarrow (a \downarrow 0) \\ 1 & \iff & \delta(f) \end{array}$$

3.

Consider the function $v : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ given by

$$v(t) = |t|, \quad |t| \leq \frac{1}{2}.$$

3.a. Note that

$$v_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| e^{-j2\pi kt} dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

so that

$$v_{-k} = v_k, \quad k = 1, 2, \ldots$$

with

$$v_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| dt = 2 \int_0^{\frac{1}{2}} t dt = \frac{1}{4}.$$

Now for each $k = 1, 2, \ldots$, we have

$$v_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} |t| e^{-j2\pi kt} dt = a_k + b_k$$

where

$$a_k := \int_0^{\frac{1}{2}} |t| e^{-j2\pi kt} dt = \int_0^{\frac{1}{2}} t e^{-j2\pi kt} dt$$

and

$$b_k := \int_{-\frac{1}{2}}^0 |t| e^{-j2\pi kt} dt = - \int_{-\frac{1}{2}}^0 t e^{-j2\pi kt} dt.$$

It is clear that

$$\begin{aligned}
b_k &= - \int_{-\frac{1}{2}}^0 t e^{-j2\pi kt} dt \\
&= - \int_{\frac{1}{2}}^0 s e^{j2\pi ks} ds \quad \text{[Change of variable } t = -s\text{]} \\
&= \int_0^{\frac{1}{2}} s e^{j2\pi ks} ds = a_k^\star.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned} a_k &= \int_0^{\frac{1}{2}} t e^{-j2\pi kt} dt \\ &= \int_0^{\frac{1}{2}} t \left(\frac{e^{-j2\pi kt}}{-j2\pi k} \right)' dt \\ &= \left[t \cdot \left(\frac{e^{-j2\pi kt}}{-j2\pi k} \right) \right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \left(\frac{e^{-j2\pi kt}}{-j2\pi k} \right) dt \\ &= \frac{1}{2} \left(\frac{e^{-j\pi k}}{-j2\pi k} \right) - \left[\frac{e^{-j2\pi kt}}{(-j2\pi k)^2} \right]_0^{\frac{1}{2}} \\ &= \frac{1}{2} \left(\frac{e^{-j\pi k}}{-j2\pi k} \right) - \frac{e^{-j\pi k} - 1}{(-j2\pi k)^2} \\ &= \frac{1}{2} \left(\frac{(-1)^k}{-j2\pi k} \right) - \frac{(-1)^k - 1}{(-j2\pi k)^2}. \end{aligned}$$

It is now immediate that

$$b_k = a_k^\star = \frac{1}{2} \left(\frac{(-1)^k}{j2\pi k} \right) - \frac{(-1)^k - 1}{(-j2\pi k)^2}$$

so that

$$v_k = a_k + b_k = -2 \frac{(-1)^k - 1}{(-j2\pi k)^2} = \frac{(-1)^k - 1}{2(\pi k)^2}.$$

Finally, for each $k = 1, 2, \ldots$, we have

$$v_k = \begin{cases} 0 & \text{if } k \text{ even} \\ -\frac{1}{(\pi k)^2} & \text{if } k \text{ odd.} \end{cases}$$

Hence,

$$\begin{aligned} v(t) &= \sum_{k=-\infty}^{\infty} v_k e^{j2\pi kt} \\ &= v_0 + \sum_{k=1}^{\infty} v_k \left(e^{j2\pi kt} + e^{-j2\pi kt} \right) \\ &= \frac{1}{4} + 2 \sum_{k=1}^{\infty} v_k \cos(2\pi kt) \\ &= \frac{1}{4} + 2 \sum_{\ell=0}^{\infty} v_{2\ell+1} \cos(2\pi(2\ell+1)t) \\ &= \frac{1}{4} - 2 \sum_{\ell=0}^{\infty} \frac{1}{(\pi(2\ell+1))^2} \cos(2\pi(2\ell+1)t) \\ &= \frac{1}{4} - \frac{2}{\pi^2} \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} \cos(2\pi(2\ell+1)t), \quad t \in \mathbb{R}. \qquad (1.6) \end{aligned}$$

3.b. By Parseval's Theorem for Fourier series we know that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |t|^2 dt = \sum_{k=-\infty}^{\infty} |v_k|^2.$$

Noting that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |t|^2 dt = \frac{1}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} 3t^2 dt = \frac{1}{3}\left(2\left(\frac{1}{2}\right)^3\right) = \frac{1}{12},$$

we conclude that $I(v) = \frac{1}{2}$.

3.c. The calculations are straightforward: Note that

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} |v_k|^2 &= |v_0|^2 + \sum_{k=-\infty}^{\infty} |v_k|^2 \\
&= \frac{1}{16} + 2\sum_{k=1}^{\infty} |v_k|^2 \\
&= \frac{1}{16} + 2\sum_{\ell=0}^{\infty} |v_{2\ell+1}|^2 \\
&= \frac{1}{16} + 2\sum_{\ell=0}^{\infty} \frac{1}{(\pi(2\ell+1))^4} \\
&= \frac{1}{16} + \frac{2}{\pi^4}\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^4} \qquad (1.7)
\end{aligned}$$

and Part **3.b** yields

$$\frac{1}{12} = \frac{1}{16} + \frac{2}{\pi^4}\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^4}.$$

Solving for π^4 we get

$$\pi^4 = 96 \cdot \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^4}.$$

4.

It is high time to compute various integrals using the properties of the Fourier transform: With the function $h : \mathbb{R} \to \mathbb{R}$ being defined by

$$h(t) = \begin{cases} 1 & \text{if } |t| \le 1 \\ 0 & \text{if } |t| > 1, \end{cases}$$

we have

$$h(t) \Longleftrightarrow H(f)$$

with

$$H(f) = 2 \cdot \frac{\sin(2\pi f)}{2\pi f}, \quad f \in \mathbb{R},$$

and by duality we conclude that

$$H(-t) \Longleftrightarrow h(f).$$

4.a. With this in mind, note that

$$\begin{aligned} 2I &= \int_{\mathbb{R}} \left(\frac{\sin t}{t} \right)^2 dt \\ &= 2\pi \int_{\mathbb{R}} \left(\frac{\sin(2\pi s)}{2\pi s} \right)^2 ds \quad [\text{Change of variable } t = 2\pi s] \\ &= \frac{\pi}{2} \int_{\mathbb{R}} \left(2 \cdot \frac{\sin(2\pi f)}{2\pi f} \right)^2 df \quad [\text{Change of variable } s = f] \\ &= \frac{\pi}{2} \int_{\mathbb{R}} |H(f)|^2 df \\ &= \frac{\pi}{2} \int_{\mathbb{R}} |h(t)|^2 dt \quad [\text{Parseval's Theorem for Fourier transforms}] \\ &= \frac{\pi}{2} \int_{-1}^{1} dt \\ &= \pi, \quad \text{hence } I = \tfrac{\pi}{2}. \end{aligned} \tag{1.8}$$

4.b. Note that

$$\begin{aligned}
I(a) &= \int_{\mathbb{R}} e^{-a|t|}.\frac{\sin t}{t}dt \\
&= 2\pi \int_{\mathbb{R}} e^{-2\pi a|s|}.\frac{\sin(2\pi s)}{2\pi s}ds \quad \text{[Change of variable } t = 2\pi s\text{]} \\
&= \pi \int_{\mathbb{R}} e^{-b|s|}.\left(2\frac{\sin(2\pi s)}{2\pi s}\right) ds \quad \text{[Set } b = 2\pi a\text{]} \\
&= \pi \int_{\mathbb{R}} e^{-b|s|} \cdot H(-s)ds \\
&= \pi \int_{\mathbb{R}} \frac{2b}{b^2 + (2\pi f)^2} \cdot h(f)df \quad \text{[By Parseval's Theorem for Fourier transforms]} \\
&= \pi \int_{-1}^{1} \frac{2b}{b^2 + (2\pi f)^2}df \\
&= \frac{2\pi}{b} \int_{-1}^{1} \frac{1}{1 + (\frac{2\pi}{b}f)^2}df \\
&= \frac{2\pi a}{b} \int_{-a^{-1}}^{a^{-1}} \frac{1}{1+x^2}dx \quad \text{[Change of variable } x = \tfrac{2\pi}{b}f = \tfrac{f}{a}\text{]} \\
&= 2\text{Arctan}\left(a^{-1}\right). \qquad (1.9)
\end{aligned}$$

4.c. Note that

$$\begin{aligned}
J(a) &= \int_0^\infty e^{-at}.\frac{\sin t}{t}dt \\
&= \int_0^\infty e^{-a|t|}.\frac{\sin t}{t}dt \\
&= \frac{1}{2}I(a) \qquad (1.10)
\end{aligned}$$

since

$$\int_{-\infty}^{0} e^{-a|t|}.\frac{\sin t}{t}dt = \int_0^\infty e^{-a|t|}.\frac{\sin t}{t}dt$$

by symmetry, and the conclusion

$$J(a) = \text{Arctan}\left(a^{-1}\right)$$

follows.

As stated in the hint to this question, there are many different ways to compute these integrals. In particular you should also be aware of the fact that if $g_1, g_2 : \mathbb{R} \to \mathbb{C}$ are finite energy signals with Fourier transforms $G_1, G_2 : \mathbb{R} \to \mathbb{C}$, then

$$\int_{\mathbb{R}} g_1(t)g_2(t)dt = (g_1 \star g_2)(0) \qquad (1.11)$$

The signal $g_1 \star g_2$ has a Fourier transform given by $G_1 \cdot G_2$, and the inverse Fourier transform yields

$$(g_1 \star g_2)(t) = \int_{\mathbb{R}} G_1(f) \cdot G_2(f) e^{j2\pi f t} df$$

so that

$$\int_{\mathbb{R}} g_1(t) g_2(t) dt = (g_1 \star g_2)(0) = \int_{\mathbb{R}} G_1(f) \cdot G_2(f) df.$$
