ENEE 420
FALL 2010
COMMUNICATIONS SYSTEMS

ANSWER KEY TO TEST # 2:

1.

1.a Fix t in $\mathbb{R}$. It is plain that

$$\begin{aligned} z(t) &= \Phi(y_{\mathrm{AM}}(t)) \\ &= aA_c\left(1+k_Am(t)\right)\cos\left(2\pi f_ct\right)+bA_c^2\left(1+k_Am(t)\right)^2\cos\left(2\pi f_ct\right)^2 \\ &= aA_c\left(1+k_Am(t)\right)\cos\left(2\pi f_ct\right) \\ &\quad +b\frac{A_c^2}{2}\left(1+k_Am(t)\right)^2\left(1+\cos\left(4\pi f_ct\right)\right) \\ &= aA_c\left(1+k_Am(t)\right)\cos\left(2\pi f_ct\right) \\ &\quad +b\frac{A_c^2}{2}\left(1+k_Am(t)\right)^2 \\ &\quad +b\frac{A_c^2}{2}\left(1+2k_Am(t)+k_A^2m(t)^2\right)\cos\left(4\pi f_ct\right). \end{aligned} \tag{1.1}$$

1.b First we feed the signal $z:\mathbb{R}\rightarrow\mathbb{R}$ through a low-pass filter with cut-off frequency $2B$, namely

$$H(f)=\begin{cases} 1 & \text{if } |f|\leq 2B \\ 0 & \text{if } |f|>2B. \end{cases}$$

Recall that m is a low-pass signal cutoff frequency B, so that the signal m^2 is also a low-pass signal but with cutoff frequency $2B$. Therefore,

$$w(t)=(h\star z)(t)=b\frac{A_c^2}{2}\left(1+k_Am(t)\right)^2, \quad t\in\mathbb{R}$$

provided the conditions

$$2B<f_c-B \quad \text{and} \quad 2B<2f_c-2B$$

hold. This amounts to

$$3B<f_c.$$

Next we use the square-rooter on w to obtain

$$\sqrt{w(t)} = \sqrt{\frac{b}{2}} A_c \left(1 + k_A m(t)\right)$$

provided

$$1 + k_A m(t) \geq 0, \quad t \in \mathbb{R}.$$

Finally a use a dc-blocker to extract $k_A m$.

Note that this approach has the advantage of avoiding multiplication by $\cos(2\pi f_c t)$ in order to achieve demodulation.

2.

2.a By standard properties of Fourier transforms we readily find

$$\begin{aligned} P(f) &= aM(f) + \sum_{k=1}^{K} \frac{a_k}{2} \left(M\left(f - \frac{k}{T}\right) + M\left(f + \frac{k}{T}\right)\right) \\ &\quad + \sum_{k=1}^{K} \frac{b_k}{2j} \left(M\left(f - \frac{k}{T}\right) - M\left(f + \frac{k}{T}\right)\right), \quad f \in \mathbb{R}. \end{aligned} \tag{1.2}$$

2.b The required filter $h : \mathbb{R} \to \mathbb{R}$ needs to satisfy the requirement $h \star p = y_{\mathrm{DSB-SC}}$, or equivalently,

$$Y_{\mathrm{DSB-SC}}(f) = H(f)P(f), \quad f \in \mathbb{R}$$

in the frequency domain, with

$$Y_{\mathrm{DSB-SC}}(f) = \frac{A_c}{2}\left(M(f - f_c) + M(f + f_c)\right), \quad f \in \mathbb{R}.$$

Upon comparing with the expression for $P(f)$, we see that h can be selected as

$$H(f) = \begin{cases} 1 & \text{if } |f - f_c| \leq B \\ 0 & \text{if } |f - f_c| > B \end{cases}$$

provided the constraints

$$f_c = \frac{k}{T} \quad \text{for some } k = 1, \ldots, K \text{ such that } a_k \neq 0 \text{ and } b_k = 0$$

and

$$2B < \frac{1}{T}$$

hold.

2.c

Yes since the periodic carrier $c : \mathbb{R} \to \mathbb{R}$ admits a Fourier representation provided similar conditions as above hold. More precisely, with Fourier series expansion

$$c(t) = \sum_k c_k e^{j2\pi \frac{k}{T} t}, \quad t \in \mathbb{R}$$

we get

$$\mathcal{F}(c \star m)(f) = \sum_k c_k M\left(f - \frac{k}{T}\right), \quad f \in \mathbb{R}$$

and the needed conditions for $y_{\text{DSB}-\text{SC}} = h \star p$ are now

$$f_c = \frac{k}{T} \quad \text{for some } k = 0, \pm 1, \ldots \text{ such that } a_k \neq 0 \text{ and } b_k = 0$$

and

$$2B < \frac{1}{T}.$$

3.

3.a We have

$$H_{\text{VSB}}(f) = \begin{cases} 0 & \text{if } 0 \leq f \leq f_c - \frac{B}{2} \\ \frac{1}{B}\left(f - f_c + \frac{B}{2}\right) & \text{if } f_c - \frac{B}{2} \leq f \leq f_c + \frac{B}{2} \\ 1 & \text{if } f_c + \frac{B}{2} \leq f \leq f_c + B \\ 0 & \text{if } f_c + B < f. \end{cases}$$

with

$$H_{\text{VSB}}(-f) = H_{\text{VSB}}(f), \quad f \geq 0.$$

From this expression for H_{VSB} (or graphically), it is easy to check that

$$H_{\text{VSB}}(f - f_c) + H_{\text{VSB}}(f + f_c) = \begin{cases} 1 & \text{if } |f| \leq B \\ 0 & \text{if } |f| > B \end{cases} \tag{1.3}$$

and the requisite condition holds.

3.b The transmission bandwidth B_T is given by

$$B_T = B + \frac{B}{2} = \frac{3B}{2}.$$

3.c By construction

$$y_{\text{VSB}} = h_{\text{VSB}} \star y_{DSB-SC}$$

so that

$$Y_{\text{VSB}}(f) = H_{\text{VSB}}(f) Y_{DSB-SC}(f), \quad f \in \mathbb{R}$$

with

$$Y_{DSB-SC}(f) = \frac{A_c}{2}\left(M(f - f_c) + M(f + f_c)\right), \quad f \in \mathbb{R}.$$

Here,

$$M(f) = \frac{A_m}{2}\left(\delta(f - f_m) + \delta(f + f_m)\right), \quad f \in \mathbb{R}$$

so that

$$
\begin{aligned}
Y_{DSB-SC}(f) &= \frac{A_c A_m}{4}\left(\delta(f-f_c-f_m)+\delta(f-f_c+f_m)\right) \\
&\quad + \frac{A_c A_m}{4}\left(\delta(f+f_c-f_m)+\delta(f+f_c+f_m)\right).
\end{aligned}
\tag{1.4}
$$

It is now plain that

$$
\begin{aligned}
& Y_{\text{VSB}}(f) \\
&= H_{\text{VSB}}(f) Y_{DSB-SC}(f) \\
&= \frac{A_c A_m}{4}\left(H_{\text{VSB}}(f_c+f_m)\delta(f-f_c-f_m)+H_{\text{VSB}}(f_c-f_m)\delta(f-f_c+f_m)\right) \\
&\quad + \frac{A_c A_m}{4}\left(H_{\text{VSB}}(-f_c+f_m)\delta(f+f_c-f_m)+H_{\text{VSB}}(-f_c-f_m)\delta(f+f_c+f_m)\right).
\end{aligned}
$$

When $\frac{1}{2} < c < 1$, we note that

$$
H_{\text{VSB}}(f_c+f_m) = H_{\text{VSB}}(-(f_c+f_m)) = 1
$$

and

$$
H_{\text{VSB}}(f_c-f_m) = H_{\text{VSB}}(-(f_c-f_m)) = 0.
$$

Therefore, we conclude that

$$
Y_{\text{VSB}}(f) = \frac{A_c A_m}{4}\left(\delta(f-(f_c+f_m))+\delta(f+(f_c+f_m))\right), \quad f \in \mathbb{R},
$$

whence

$$
y_{\text{VSB}}(t) = \frac{A_c A_m}{2}\cos\left(2\pi(f_c+f_m)t\right), \quad t \in \mathbb{R}.
$$

3.d When $0 < c < \frac{1}{2}$, we now have

$$
H_{\text{VSB}}(f_c+f_m) = H_{\text{VSB}}(-(f_c+f_m)) = \frac{1}{B}\left(f_m+\frac{B}{2}\right) = c+\frac{1}{2}
$$

and

$$
H_{\text{VSB}}(f_c-f_m) = H_{\text{VSB}}(-(f_c-f_m)) = \frac{1}{B}\left(-f_m+\frac{B}{2}\right) = -c+\frac{1}{2}
$$

whence

$$
\begin{aligned}
& Y_{\text{VSB}}(f) \\
&= \frac{A_c A_m}{4}\left(\left(c+\frac{1}{2}\right)\delta(f-f_c-f_m)+\left(-c+\frac{1}{2}\right)\delta(f-f_c+f_m)\right) \\
&\quad + \frac{A_c A_m}{4}\left(\left(-c+\frac{1}{2}\right)\delta(f+f_c-f_m)+\left(c+\frac{1}{2}\right)\delta(f+f_c+f_m)\right).
\end{aligned}
$$

It is now easy to see that

$$
y_{\text{VSB}}(t) = \frac{A_c A_m}{2}\left(\left(c+\frac{1}{2}\right)\cos\left(2\pi(f_c+f_m)t\right)+\left(-c+\frac{1}{2}\right)\cos\left(2\pi(f_c-f_m)t\right)\right)
$$

for each t in $\mathbb{R}$.

4.

4.a The signal $t \to \varepsilon(t)$ is periodic with period T, and we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left|e^{jk_P m(t)}\right|^2 dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = T.$$

Thus, the signal $t \to \varepsilon(t)$ admits a Fourier series representation of the form

$$\varepsilon(t) = \sum_k c_k e^{j2\pi \frac{k}{T} t}, \quad t \in \mathbb{R}$$

with Fourier coefficients

$$c_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{jk_P m(t)} e^{-j2\pi \frac{k}{T} t} dt, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Now, for a given $k = 0, \pm 1, \pm 2, \ldots$, we get

$$\begin{aligned} c_k &= \frac{1}{T} \int_{-\frac{T}{2}}^{0} e^{jk_P m(t)} e^{-j2\pi \frac{k}{T} t} dt + \frac{1}{T} \int_{0}^{\frac{T}{2}} e^{jk_P m(t)} e^{-j2\pi \frac{k}{T} t} dt \\ &= \frac{1}{T} \int_{-\frac{T}{2}}^{0} e^{-jk_P} e^{-j2\pi \frac{k}{T} t} dt + \frac{1}{T} \int_{0}^{\frac{T}{2}} e^{jk_P} e^{-j2\pi \frac{k}{T} t} dt \\ &= \frac{1}{T} \int_{0}^{\frac{T}{2}} e^{-jk_P} e^{j2\pi \frac{k}{T} t} dt + \frac{1}{T} \int_{0}^{\frac{T}{2}} e^{jk_P} e^{-j2\pi \frac{k}{T} t} dt. \end{aligned} \tag{1.5}$$

For $k = 0$,

$$c_0 = \frac{1}{2} \left(e^{jk_P} + e^{-jk_P}\right) = \cos\left(k_P\right).$$

For $k = \pm 1, \pm 2, \ldots$,

$$\begin{aligned} c_k &= \frac{e^{-jk_P}}{T} \left(\frac{e^{j\pi k} - 1}{j2\pi \frac{k}{T}} \right) + \frac{e^{jk_P}}{T} \left(\frac{e^{-j\pi k} - 1}{-j2\pi \frac{k}{T}} \right) \\ &= e^{-jk_P} \left(\frac{e^{j\pi k} - 1}{j2\pi k} \right) + e^{jk_P} \left(\frac{e^{-j\pi k} - 1}{-j2\pi k} \right) \\ &= e^{-jk_P} \left(\frac{(-1)^k - 1}{j2\pi k} \right) + e^{jk_P} \left(\frac{(-1)^k - 1}{-j2\pi k} \right) \\ &= \left(\frac{(-1)^k - 1}{\pi k} \right) \cdot \left(\frac{e^{-jk_P} - e^{jk_P}}{2j} \right) \\ &= -\left(\frac{(-1)^k - 1}{\pi k} \right) \sin\left(k_P\right) \end{aligned} \tag{1.6}$$

As a result,

$$\begin{aligned}
\varepsilon(t) &= \cos(k_P) - \sin(k_P) \sum_{k\neq 0} \left(\frac{(-1)^k - 1}{\pi k} \right) e^{j2\pi \frac{k}{T} t} \\
&= \cos(k_P) - \sin(k_P) \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{\pi k} \right) \left(e^{j2\pi \frac{k}{T} t} - e^{-j2\pi \frac{k}{T} t} \right) \\
&= \cos(k_P) - 2j \sin(k_P) \sum_{k=1}^{\infty} \left(\frac{(-1)^k - 1}{\pi k} \right) \sin\left(2\pi \frac{k}{T} t \right) \\
&= \cos(k_P) + j \frac{4\sin(k_P)}{\pi} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \sin\left(2\pi \frac{2\ell+1}{T} t \right). \qquad (1.7)
\end{aligned}$$

4.b The PM signal $y_{\mathrm{PM}} : \mathbb{R} \to \mathbb{R}$ can be evaluated as follows: For each t in $\mathbb{R}$, we get

$$\begin{aligned}
y_{\mathrm{PM}}(t) &= A_c \cos(2\pi f_c t + k_P m(t)) \\
&= A_c \mathrm{Re}\left(e^{j(2\pi f_c t + k_P m(t))} \right) \\
&= A_c \mathrm{Re}\left(e^{j2\pi f_c t} \varepsilon(t) \right) \\
&= A_c \mathrm{Re}\left(e^{j2\pi f_c t} \left(\cos(k_P) + j \frac{4\sin(k_P)}{\pi} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \sin\left(2\pi \frac{2\ell+1}{T} t \right) \right) \right) \\
&= A_c \cos(k_P) \cos(2\pi f_c t) \\
&\quad - \frac{4A_c}{\pi} \sin(k_P) \sin(2\pi f_c t) \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \sin\left(2\pi \frac{2\ell+1}{T} t \right) \\
&= A_c \cos(k_P) \cos(2\pi f_c t) \\
&\quad - \frac{2A_c}{\pi} \sin(k_P) \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \cos\left(2\pi \left(f_c - \frac{2\ell+1}{T} \right) t \right) \\
&\quad + \frac{2A_c}{\pi} \sin(k_P) \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \cos\left(2\pi \left(f_c + \frac{2\ell+1}{T} \right) t \right). \qquad (1.8)
\end{aligned}$$

4.c The PM signal $y_{\mathrm{PM}} : \mathbb{R} \to \mathbb{R}$ has frequency content at the discrete frequencies

$$f = \pm f_c \quad \text{and} \quad f = \pm\left(f_c \pm \frac{2\ell+1}{T} \right), \quad \ell = 0, 1, \ldots$$

The power of the PM signal $y_{\mathrm{PM}} : \mathbb{R} \to \mathbb{R}$ can now be computed in the usual manner:

$$\begin{aligned}
\mathcal{P}_{y_{\mathrm{PM}}} &= \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} |y_{\mathrm{PM}}(t)|^2 \, dt \\
&= A_c^2 \cdot \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} |\cos(2\pi f_c t + k_P m(t))|^2 \, dt \\
&= \frac{A_c^2}{2} \left(1 + \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} \cos(4\pi f_c t + 2k_P m(t)) \, dt \right) \\
&= \frac{A_c^2}{2} \qquad (1.9)
\end{aligned}$$

for it is easy to see that

$$\lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} \cos\left(4\pi f_c t + 2k_P m(t)\right) dt = 0$$

by a symmetry argument that uses the form of m.

Another approach is to use your answer in Part **4.c**:

$$\begin{aligned}
\mathcal{P}_{y_{\rm PM}} &= \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} |y_{\rm PM}(t)|^2 \, dt \\
&= \frac{A_c^2}{2} \cos\left(k_P\right)^2 + \frac{2A_c^2}{\pi^2} \sin\left(k_P\right)^2 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} \\
&\quad + \frac{2A_c^2}{\pi^2} \sin\left(k_P\right)^2 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} \\
&= \frac{A_c^2}{2} \cos\left(k_P\right)^2 + \frac{4A_c^2}{\pi^2} \sin\left(k_P\right)^2 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} \\
&= \frac{A_c^2}{2}. \qquad (1.10)
\end{aligned}$$

This last fact follows from the well-known identity[1]

$$\sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = \frac{\pi^2}{8}. \qquad (1.11)$$

Indeed (1.10) holds if

$$\cos\left(k_P\right)^2 + \frac{8}{\pi^2} \sin\left(k_P\right)^2 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = 1,$$

and this condition can be rewritten as

$$\frac{8}{\pi^2} \sin\left(k_P\right)^2 \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)^2} = 1 - \cos\left(k_P\right)^2 = \sin\left(k_P\right)^2,$$

and (1.11) validates it!

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[1] This is also a consequence of Fourier series analysis with the help of Parseval's Theorem. You were not expected to know it.