ENEE 420
FALL 2012
COMMUNICATIONS SYSTEMS

ANGLE MODULATION

Throughout, we consider the information-bearing signal $m : \mathbb{R} \to \mathbb{R}$. Its Fourier transform is given by

$$M(f) := \int_{\mathbb{R}} m(t) e^{-j2\pi f t} dt, \quad f \in \mathbb{R}.$$

Frequency modulation

The FM waveform $s_{\rm FM} : \mathbb{R} \to \mathbb{R}$ associated with the information-bearing signal m is given by

$$s_{\rm FM}(t) = A_c \cos\left(\theta_{\rm FM}(t)\right), \quad t \in \mathbb{R}$$

with

$$\theta_{\rm FM}(t) = 2\pi f_c t + 2\pi k_F \int_0^t m(r) dr, \quad t \in \mathbb{R}.$$

Phase modulation

The PM waveform $s_{\rm PM} : \mathbb{R} \to \mathbb{R}$ associated with the information-bearing signal m is given by

$$s_{\rm PM}(t) = A_c \cos\left(\theta_{\rm PM}(t)\right), \quad t \in \mathbb{R}$$

with

$$\theta_{\rm PM}(t) = 2\pi f_c t + k_P m(t), \quad t \in \mathbb{R}.$$

Single-tone modulating signals

In an attempt to understand how the spectrum of angle-modulated signals is shaped by that of the modulating signal, we consider the simple case of a single-tone modulating signal $m : \mathbb{R} \to \mathbb{R}$, say

$$m(t) = A_m \cos(2\pi f_m t), \quad t \in \mathbb{R}$$

with amplitude $A_m > 0$ and frequency $f_m > 0$. In that case, we note that

$$\begin{aligned}\theta_{\rm FM}(t) &= 2\pi f_c t + 2\pi k_F \int_0^t A_m \cos(2\pi f_m r)\, dr \\ &= 2\pi f_c t + 2\pi \frac{k_F A_m}{2\pi f_m} \sin(2\pi f_m t) \\ &= 2\pi f_c t + \frac{k_F A_m}{f_m} \sin(2\pi f_m t) \\ &= 2\pi f_c t + \beta \sin(2\pi f_m t), \quad t \in \mathbb{R} \qquad (1)\end{aligned}$$

where

$$\beta := \frac{\Delta f}{f_m} \quad \text{and} \quad \Delta f := k_F A_m.$$

Next,

$$\begin{aligned}\cos(\theta_{\rm FM}(t)) &= \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= \Re\left(e^{j2\pi f_c t} e^{j\beta \sin(2\pi f_m t)}\right), \quad t \in \mathbb{R}. \qquad (2)\end{aligned}$$

The function $t \to e^{j\beta \sin(2\pi f_m t)}$ being continuous and periodic with period $T_m = \frac{1}{f_m}$, it admits the Fourier series representation

$$e^{j\beta \sin(2\pi f_m t)} = \sum_k c_k e^{j2\pi k f_m t}, \quad t \in \mathbb{R}$$

with

$$c_k = \frac{1}{T_m} \int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi k f_m t} dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

Now fix $k = 0, \pm 1, \pm 2, \ldots$. Upon making the change of variable $x = 2\pi f_m t$, we get

$$\begin{aligned}c_k &= \frac{1}{T_m} \int_{-\frac{T_m}{2}}^{\frac{T_m}{2}} e^{j\beta \sin(2\pi f_m t)} e^{-j2\pi k f_m t} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(x) - kx)} dx \\ &= J_k(\beta) \qquad (3)\end{aligned}$$

where

$$J_k(\beta) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin(x)-kx)}dx, \quad \beta \in \mathbb{R}$$

defines the k^{th} order Bessel function of the first kind.

Substituting we find

$$e^{j\beta\sin(2\pi f_m t)} = \sum_k J_k(\beta)e^{j2\pi k f_m t}, \quad t \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} A_c\cos(\theta_{\mathrm{FM}}(t)) &= A_c\Re\left(e^{j2\pi f_c t}e^{j\beta\sin(2\pi f_m t)}\right) \\ &= A_c\Re\left(e^{j2\pi f_c t}\sum_k J_k(\beta)e^{j2\pi k f_m t}\right) \\ &= A_c\sum_k J_k(\beta)\Re\left(e^{j2\pi f_c t}e^{j2\pi k f_m t}\right) \\ &= A_c\sum_k J_k(\beta)\cos\left(2\pi\left(f_c + k f_m\right)t\right), \quad t \in \mathbb{R}. \end{aligned} \tag{4}$$

In the frequency domain this last relationship becomes

$$S_{\mathrm{FM}}(f) = \frac{A_c}{2}\sum_k J_k(\beta)\left(\delta(f-(f_c+kf_m)) + \delta(f+(f_c+kf_m))\right) \tag{5}$$

for all f in $\mathbb{R}$. Thus, although the single-tone signal m has frequency content *only* at the frequencies $f = \pm f_m$, the corresponding FM wave has *infinite* bandwidth since it displays frequency content at the countably infinite set of frequencies

$$f = \pm(f_c + kf_m), \quad k = 0, \pm 1, \ldots .$$

Narrow-band vs wide-band FM

Using elementary trigonometric formulae, we observe

$$\begin{aligned} s_{\mathrm{FM}}(t) &= A_c\cos(\theta_{\mathrm{FM}}(t)) \\ &= A_c\cos\left(2\pi f_c t + 2\pi k_F\int_0^t m(r)dr\right) \\ &= A_c\cos(2\pi f_c t)\cos\left(2\pi k_F\int_0^t m(r)dr\right) \\ &\quad - A_c\sin(2\pi f_c t)\sin\left(2\pi k_F\int_0^t m(r)dr\right), \quad t \in \mathbb{R} \end{aligned} \tag{6}$$

Narrow-band FM is characterized by

$$2\pi k_F \left| \int_0^t m(r)dr \right| \ll 1, \quad t \in \mathbb{R} \tag{7}$$

in which case

$$\sin\left(2\pi k_F \int_0^t m(r)dr\right) \simeq 2\pi k_F \int_0^t m(r)dr$$

and

$$\cos\left(2\pi k_F \int_0^t m(r)dr\right) \simeq 1$$

for all t in $\mathbb{R}$. Therefore, we have the approximation

$$s_{\mathrm{FM}}(t) \simeq s_{\mathrm{NB-FM}}(t), \quad t \in \mathbb{R} \tag{8}$$

where the narrow-band FM signal $s_{\mathrm{NB-FM}} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned} s_{\mathrm{NB-FM}}(t) &= A_c \cos(2\pi f_c t) \\ &\quad - A_c \sin(2\pi f_c t)\left(2\pi k_F \int_0^t m(r)dr\right), \quad t \in \mathbb{R}. \end{aligned} \tag{9}$$

In other words, when condition (7) holds, the FM waveform s_{FM} is well approximated by $s_{\mathrm{NB-FM}}$ and therefore can be replaced by it. The advantage of doing so is that the signal $s_{\mathrm{NB-FM}}$ is AM-like in its structure and can be generated easily according to techniques developed for amplitude modulation. *Wide-band FM* arises when the condition (7) fails to hold.

Carson's formula

The realization that the spectrum of s_{FM} has infinite extent leads to the following practical concern: How much bandwidth is needed to transmit s_{FM} without too much distortion?

One answer to this question was given by Carson, and is summarized in the formula that carries his name: Carson's formula states that the transmission bandwidth B_T of the FM wave associated with the single-tone signal m should be set to

$$\begin{aligned} B_{T,\mathrm{Carson}} &:= 2f_m + 2\Delta f \\ &= 2f_m(1+\beta) \end{aligned} \tag{10}$$

since $\Delta f = f_m \beta$ by definition.

One way to generalize Carson's bandwidth formula could proceed by *formally* giving the quantities f_m and β interpretations which do not rely on the specific form of the information-bearing signal m. We do this as follows:

In the single-tone case, the frequency f_m can be interpreted as the cutoff frequency of the signal – In other words, f_m is the bandwidth of the signal. On the other hand, Δf can be viewed as describing the largest possible excursion of the instantaneous frequency from f_c: Indeed, the instantaneous frequency of the FM wave at time t is given by

$$\frac{1}{2\pi}\frac{d}{dt}\theta_{\mathrm{FM}}(t) = f_c + k_F A_m \cos\left(2\pi f_m t\right)$$

and the corresponding deviation in instantaneous frequency at time t is simply

$$\frac{1}{2\pi}\frac{d}{dt}\theta_{\mathrm{FM}}(t) - f_c = k_F A_m \cos\left(2\pi f_m t\right).$$

Therefore, the maximal deviation from f_c is given by

$$\sup\left(\left|k_F A_m \cos\left(2\pi f_m t\right)\right|, \quad t \in \mathbb{R}\right) = k_F A_m = \Delta f.$$

Now consider an information bearing signal which is bandlimited with cutoff frequency $W > 0$. With the discussion for the single-tone modulating signal in mind, it is natural to replace in Carson's formula f_m by W and Δf by D with

$$D := \sup\left(k_F \left|m(t)\right|, \ t \in \mathbb{R}\right).$$

This suggests the approximation

$$B_T \simeq B_{T,\mathrm{Carson}}$$

with

$$\begin{aligned} B_{T,\mathrm{Carson}} &:= 2W + 2D \\ &= 2W\left(1+\beta\right) \end{aligned} \tag{11}$$

where β is defined as

$$\beta := \frac{D}{W} = \frac{\sup\left(k_F \left|m(t)\right|, \ t \in \mathbb{R}\right)}{W}.$$

At this point, you may feel that the generalized Carson's formula discussed above is simply a formal expression without much practical grounding. We now show through an approximation argument (see below) that the bandwidth as given by $B_{T,\text{Carson}}$ is indeed meaningful from an engineering point of view.

The basic idea is to characterize the spectrum of the FM wave associated with a *sampled* version of the information-bearing signal. Thus, fix $T > 0$. We approximate the information-bearing signal $m : \mathbb{R} \to \mathbb{R}$ by the staircase approximation $m^\star_T : \mathbb{R} \to \mathbb{R}$ given by

$$m^\star_T(t) = m(kT), \quad kT \le t < (k+1)T$$

with $k = 0, \pm 1, \ldots$. We then replace $\theta_{\text{FM}} : \mathbb{R} \to \mathbb{R}$ as defined above by $\theta^\star_{\text{FM},T} : \mathbb{R} \to \mathbb{R}$ given by

$$\theta^\star_{\text{FM},T}(t) = 2\pi f_c t + 2\pi k_F \int_0^t m^\star_T(r)dr, \quad t \in \mathbb{R}$$

and write

$$s^\star_{\text{FM},T}(t) = A_c \cos\left(\theta^\star_{\text{FM},T}(t)\right), \quad t \in \mathbb{R}.$$

Fix f in $\mathbb{R}$. Note that

$$\begin{aligned} S^\star_{\text{FM},T}(f) &= \int_{\mathbb{R}} A_c \cos\left(\theta^\star_{\text{FM},T}(t)\right) e^{-j2\pi f t} dt \\ &= A_c \sum_k \int_{kT}^{(k+1)T} \cos\left(\theta^\star_{\text{FM},T}(t)\right) e^{-j2\pi f t} dt. \end{aligned} \tag{12}$$

Now, for $k = 0, 1, \ldots$, with $kT \le t < (k+1)T$, we have

$$\begin{aligned} \theta^\star_{\text{FM},T}(t) &= 2\pi f_c t + 2\pi k_F \int_0^t m^\star_T(r)dr \\ &= 2\pi f_c t + 2\pi k_F \left(T \sum_{\ell=0}^{k-1} m(\ell T) + m(kT)(t - kT) \right) \\ &= 2\pi(f_c + k_F m(kT))(t - kT) + 2\pi T \left(k f_c + k_F \sum_{\ell=0}^{k-1} m(\ell T) \right) \\ &= 2\pi(f_c + k_F m(kT))(t - kT) + 2\pi \gamma_k T \end{aligned} \tag{13}$$

where we have set

$$\gamma_k := kf_c + k_F \left(\sum_{\ell=0}^{k-1} m(\ell T) \right).$$

Direct substitution yields

$$\begin{aligned} & \int_{kT}^{(k+1)T} \cos\left(\theta^\star_{\mathrm{FM},T}(t)\right) e^{-j2\pi f t} dt \\ = & \int_{kT}^{(k+1)T} \cos\left(2\pi(f_c + k_F m(kT))(t - kT) + 2\pi\gamma_k T\right) e^{-j2\pi f t} dt \\ = & e^{-j2\pi k f T} \cdot \int_0^T \cos\left(2\pi(f_c + k_F m(kT))\tau + 2\pi\gamma_k T\right) e^{-j2\pi f \tau} d\tau. \end{aligned} \tag{14}$$

To evaluate this last integral, we note that

$$\begin{aligned} & \int_0^T e^{\pm j2\pi((f_c + k_F m(kT))\tau + \gamma_k T)} e^{-j2\pi f\tau} d\tau \\ = & e^{\pm j2\pi\gamma_k T} \int_0^T e^{j2\pi(\pm(f_c + k_F m(kT)) - f)\tau} d\tau \\ = & e^{\pm j2\pi\gamma_k T} \cdot \frac{e^{j2\pi(\pm(f_c + k_F m(kT)) - f)T} - 1}{j2\pi\left(\pm(f_c + k_F m(kT)) - f\right)} \\ = & a_k^\pm(f) \frac{\sin\left(\pi\left(\pm(f_c + k_F m(kT)) - f\right)T\right)}{\pi\left(\pm(f_c + k_F m(kT)) - f\right)} \\ = & a_k^\pm(f) \frac{\sin\left(\pi\left(f \mp (f_c + k_F m(kT))\right)T\right)}{\pi\left(f \mp (f_c + k_F m(kT))\right)} \end{aligned} \tag{15}$$

with

$$a_k^\pm(f) = e^{j2\pi\delta_k^\pm(f)T}$$

where

$$\delta_k^\pm(f) = \pm\gamma_k + \frac{1}{2}\left(\pm(f_c + k_F m(kT)) - f\right).$$

Recall that the sinc function $\mathrm{sinc} : \mathbb{R} \to \mathbb{R}$ is given by

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbb{R}.$$

Therefore, for each $k = 0, 1, \ldots$, we have

$$\int_{kT}^{(k+1)T} \cos\left(\theta^\star_{\mathrm{FM},T}(t)\right) e^{-j2\pi f t} dt$$

$$
\begin{aligned}
&= \frac{1}{2}a_k^+(f)\frac{\sin\left(\pi\left(f-(f_c+k_F m(kT))\right)T\right)}{\pi\left(f-(f_c+k_F m(kT))\right)} \\
&\quad +\frac{1}{2}a_k^-(f)\frac{\sin\left(\pi\left(f+(f_c+k_F m(kT))\right)T\right)}{\pi\left(f+(f_c+k_F m(kT))\right)} \\
&= \frac{1}{2}a_k^+(f)\cdot\mathrm{sinc}\left(\left(f-(f_c+k_F m(kT))\right)T\right) \\
&\quad +\frac{1}{2}a_k^-(f)\cdot\mathrm{sinc}\left(\left(f+(f_c+k_F m(kT))\right)T\right),
\end{aligned} \tag{16}
$$

and we can conclude

$$
\begin{aligned}
&\int_0^\infty \cos\left(\theta^\star_{\mathrm{FM},T}(t)\right)e^{-j2\pi ft}dt \\
&= \frac{1}{2}\sum_{k=0}^\infty a_k^+(f)\cdot\mathrm{sinc}\left(\left(f-(f_c+k_F m(kT))\right)T\right) \\
&\quad +\frac{1}{2}\sum_{k=0}^\infty a_k^-(f)\cdot\mathrm{sinc}\left(\left(f+(f_c+k_F m(kT))\right)T\right).
\end{aligned} \tag{17}
$$

The zeroes of the sinc function occur at $x = \pm\ell$, $\ell = 1, 2, \ldots$, and its main lobe occupies the interval $[-1, 1]$. As a result, for each $k = 0, 1, \ldots$, the main contribution of the term

$$\frac{1}{2}a_k^\pm(f)\cdot\mathrm{sinc}\left(\left(f\mp(f_c+k_F m(kT))\right)T\right)$$

is taking place on an an interval centered at

$$\pm(f_c+k_F m(kT))$$

and of length $2/T$, namely

$$\left[\pm(f_c+k_F m(kT))-\frac{1}{T}, \pm(f_c+k_F m(kT))+\frac{1}{T}\right].$$

Similar arguments could be made for the case $k = -1, -2, \ldots$ and would lead to a similar expression for

$$\int_{-\infty}^0 \cos\left(\theta^\star_{\mathrm{FM},T}(t)\right)e^{-j2\pi ft}dt, \quad f\in\mathbb{R}.$$

The discussion suggests that most of the spectral content is contained in the interval

$$\left[\pm(f_c - D) - \frac{1}{T}, \pm(f_c + D) + \frac{1}{T}\right]$$

since

$$|k_F m(kT)| \leq D, \quad k = 0, \pm 1, \ldots$$

by the definition of D. This leads to estimating the transmission bandwidth of $s^\star_{\mathrm{FM},T}$ as being

$$B_T \simeq 2D + \frac{2}{T}.$$

If we sample at the Nyquist rate, that is $T = \frac{1}{2W}$, then the information contained in m is recoverable from $m^\star_T$, and the transmission bandwidths of their corresponding FM waveforms should be commensurate. In short,

$$B^\star = 2D + 4W$$

is expected to provide a reasonably good approximation to B_T. Note that

$$B^\star = 2D + 2W + 2W = B_{T,\mathrm{Carson}} + 2W$$

so that this argument provides an approximation to the transmissison bandwith of the FM wave s_{FM} which is more conservative than the one provide by Carson's formula. This can be traced to the fact that the approximation is based on a sampling argument.

Immunity of angle modulation to non-linearities

Consider a non-linear device $\varphi : \mathbb{R} \to \mathbb{R}$ of the form

$$\varphi(x) = \sum_{m=1}^{M} a_m x^m, \quad x \in \mathbb{R}$$

for some integer $M \geq 2$ and assume $a_M \neq 0$.

For each t in $\mathbb{R}$, we note that

$$\varphi(s_{\mathrm{FM}}(t)) = \sum_{m=1}^{M} a_m \left(A_c \cos(\theta_{\mathrm{FM}}(t))\right)^m$$

$$\begin{aligned} &= \sum_{m=1}^{M} a_m A_c^m \left(\cos(\theta_{\rm FM}(t))\right)^m \\ &= \sum_{m=1}^{M} a_m A_c^m \left(\sum_{k=0}^{m} a_{m,k} \cos(k\theta_{\rm FM}(t)) \right) \end{aligned} \tag{18}$$

as we invoke Lemma 0.1 in the last step. Interchanging the order of summation we conclude that

$$\begin{aligned} \varphi(s_{\rm FM}(t)) &= \sum_{m=1}^{M} a_m A_c^m a_{m,0} \\ &\quad + \sum_{k=1}^{M} \left(\sum_{m=k}^{M} a_m A_c^m a_{m,k} \right) \cos(k\theta_{\rm FM}(t)) \\ &= \sum_{\ell=0}^{M} B_{M,\ell} \cos(\ell\theta_{\rm FM}(t)) \end{aligned} \tag{19}$$

with

$$B_{M,\ell} = \begin{cases} \sum_{m=1}^{M} a_m A_c^m a_{m,0} & \text{if } \ell = 0 \\ \\ \sum_{m=\ell}^{M} a_m A_c^m a_{m,\ell} & \text{if } \ell = 1, \ldots, M. \end{cases} \tag{20}$$

For each $\ell = 1, \ldots, M$, the signal $t \to \cos(\ell\theta_{\rm FM}(t))$ is the FM waveform at carrier frequency ℓf_c generated by the signal $t \to \ell m(t)$. According to the generalized Carson's rule, for all practical intent, we can view this signal as a bandpass signal whose (transmission) bandwidth B_ℓ is given by

$$B_\ell = 2(W + D_\ell) \quad \text{with} \quad D_\ell = \ell D$$

since

$$\begin{aligned} D_\ell &= \sup\left(k_F \left|\ell m(t)\right|, \ t \in \mathbb{R}\right) \\ &= \ell \sup\left(k_F \left|m(t)\right|, \ t \in \mathbb{R}\right) = \ell D. \end{aligned} \tag{21}$$

Under the appropriate conditions each of the components $t \to \cos(\ell\theta_{\rm FM}(t))$ can be extracted from $\varphi(s_{\rm FM})$ by means of bandpass filtering. For instance, to recover $s_{\rm FM}$ from $\varphi(s_{\rm FM})$ we pass the latter through a bandpass filter centered at f_c with bandwidth B_1 such that

$$f_c + \frac{B_1}{2} < 2f_c - \frac{B_2}{2}.$$

This is equivalent to

$$f_c + (W + D) < 2f_c - (W + 2D),$$

and requires that the condition

$$2W + 3D < f_c$$

holds.

Similar arguments can be given for extracting $t \to \cos(\ell \theta_{\rm FM}(t))$ by means of bandpass filtering for $\ell = 2, \ldots . M$.

Generating FM signals

Direct method Using a voltage-controlled oscillator (VCO)

Indirect method of Armstrong We seek to generate the FM signal $s_{\rm FM} : \mathbb{R} \to \mathbb{R}$ associated with the information-bearing signal m, say

$$s_{\rm FM}(t) = A_c \cos\left(\theta_{\rm FM}(t)\right), \quad t \in \mathbb{R}$$

with

$$\theta_{\rm FM}(t) = 2\pi f_c t + 2\pi k_F \int_0^t m(r)dr, \quad t \in \mathbb{R}$$

for some given $k_F > 0$. We are in the situation when the condition (7) fails to hold for the choice of k_F so that $s_{\rm NB-FM}$ is not a good approximation to the desired FM signal $s_{\rm FM}$.

We begin by writing $k_F = M k_F^\star$ for some positive integer M, so that the condition (7) now holds for $k_F^\star$, namely

$$2\pi k_F^\star \left| \int_0^t m(r)dr \right| \ll 1, \quad t \in \mathbb{R}.$$

Under this condition the FM signal $s_{\rm FM}^\star : \mathbb{R} \to \mathbb{R}$ given by

$$s_{\rm FM}^\star = A_c \cos\left(2\pi f_c t + 2\pi k_F^\star \int_0^t m(s)ds\right), \quad t \in \mathbb{R}$$

can be well aprroximated by the narrow-band FM signal $s^\star_{\rm NB-FM} : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} s^\star_{\rm NB-FM}(t) &= A_c \cos\left(2\pi f_c t\right) \\ &\quad - A_c \sin\left(2\pi f_c t\right) \left(2\pi k^\star_F \int_0^t m(r) dr \right), \quad t \in \mathbb{R}. \end{aligned} \tag{22}$$

Next, the narrow-band FM signal $s^\star_{\rm NB-FM} : \mathbb{R} \to \mathbb{R}$ is converted to the desired wide-band FM signal as follows: Consider a non-linear device $\varphi : \mathbb{R} \to \mathbb{R}$ of the form

$$\varphi(x) = \sum_{m=1}^{M} a_m x^m, \quad x \in \mathbb{R}$$

with $a_M \neq 0$.

For each t in $\mathbb{R}$, with

$$\theta^\star_{\rm FM}(t) = 2\pi f_c t + 2\pi k^\star_F \int_0^t m(r) dr,$$

we note from (19)-(20) that

$$\varphi(s^\star_{\rm FM}(t)) = \sum_{\ell=0}^{M} B_{M,\ell} \cos(\ell \theta^\star_{\rm FM}(t)) \tag{23}$$

with the coefficients as given by (20).

By the same arguments as given earlier in the discussion of immunity of angle modulation to non-linearities, we can extract the signal $t \to \cos(M\theta^\star_{\rm FM}(t))$ by feeding the signal $t \to \varphi(s^\star_{\rm FM}(t))$ through a bandpass filter with center frequency Mf_c and bandwidth $B^\star_M$ given by

$$B^\star_M = 2(W + D^\star_M)$$

where for each $\ell = 1, 2, \ldots$, we have

$$\begin{aligned} D^\star_\ell &= \sup\left(k^\star_F \left|\ell m(t)\right|, \ t \in \mathbb{R}\right) \\ &= \ell \sup\left(k^\star_F \left|m(t)\right|, \ t \in \mathbb{R}\right) \\ &= \frac{\ell}{M} \sup\left(k_F \left|m(t)\right|, \ t \in \mathbb{R}\right). \\ &= \frac{\ell}{M} \cdot D. \end{aligned} \tag{24}$$

As a result,

$$B^\star_M = 2(W + D^\star_M) = 2(W + D)$$

as should be expected!

Properties of Bessel functions

0. For each $k = 0, \pm 1, \ldots$ and every β in $\mathbb{R}$, $J_k(\beta)$ is an element of $\mathbb{R}$.

Proof. Fix $k = 0, \pm 1, \ldots$ and β in $\mathbb{R}$. Note that

$$\begin{aligned} J_k(\beta)^\star &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - kx)} dx \right)^\star \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(\beta \sin x - kx)} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(-\beta \sin x + kx)} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(-x) - k(-x))} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin y - ky)} dy \\ &= J_k(\beta), \end{aligned} \tag{25}$$

whence $J_k(\beta)$ is an element of $\mathbb{R}$. ■

1. For each $k = 0, 1, \ldots$, we have

$$J_{-k}(\beta) = (-1)^k J_k(\beta), \quad \beta \in \mathbb{R}.$$

Proof. Fix $k = 0, 1, \ldots$ and β in $\mathbb{R}$. Using the change of variable $y = \pi - x$ we find

$$\begin{aligned} J_{-k}(\beta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x + kx)} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin(\pi - y) + k(\pi - y))} dy \\ &= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin y - y))} dy \right) \cdot e^{jk\pi} \\ &= (-1)^k J_k(\beta) \end{aligned} \tag{26}$$

since $e^{jk\pi} = (-1)^k$. ■

2. For each $k = 0, 1, \ldots$, we have

$$J_k(-\beta) = (-1)^k J_k(\beta), \quad \beta \geq 0.$$

Proof. Fix $k = 0, 1, \ldots$ and $\beta \geq 0$. We note that

$$\begin{aligned} J_k(-\beta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(-\beta \sin y - ky)} dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin(-y) + k(-y))} dy \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x + kx)} dx \\ &= J_{-k}(\beta) \end{aligned} \tag{27}$$

and the conclusion follows by Fact **1**. ■

3. We have

$$J_0(\beta) = 1 + O(\beta) \quad (\beta \to 0).$$

Proof. Fix β in $\mathbb{R}$. From the definitions we see that

$$\begin{aligned} J_0(\beta) - 1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^{j\beta \sin x} - 1\right) dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\int_0^{\beta \sin x} j e^{jt} dt\right) dx \end{aligned} \tag{28}$$

so that

$$\begin{aligned} |J_0(\beta) - 1| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\int_0^{\beta \sin x} j e^{jt} dt\right| dx \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\int_0^{|\beta \sin x|} \left|j e^{jt}\right| dt\right| dx \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\beta| |\sin x| dx \\ &\leq |\beta| \end{aligned} \tag{29}$$

and the conclusion follows. ■

4. We have

$$J_1(\beta) = \frac{\beta}{2}(1 + o(1)) \quad (\beta \to 0).$$

5. For each $\ell = 0, 1, \ldots$ we have

$$J_\ell(\beta) = \frac{\beta^\ell}{2^\ell \ell!}(1 + o(1)) \quad (\beta \to 0).$$

6. For each β in $\mathbb{R}$, we have

$$\sum_\ell |J_\ell(\beta)|^2 = 1.$$

Proof. For each β in $\mathbb{R}$, the function $x \to e^{j \sin x}$ is periodic with period 2π and therefore admits a Fourier series representation. It is a simple matter to see that

$$e^{j \sin x} = \sum_\ell J_k(\beta) e^{j\ell x}$$

and by Parseval's Theorem we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |e^{j\beta \sin x}|^2 dx = \sum_\ell |J_\ell(\beta)|^2.$$

The conclusion follows from the fact that

$$|e^{j\beta \sin x}|^2 = 1, \quad x \in \mathbb{R}.$$

■

On powers of $\cos\theta$

Given is θ in $\mathbb{R}$. We are interested in understanding how to compute

$$(\cos\theta)^m, \quad m = 1, 2, \ldots$$

We shall repeatedly use the trigonometric identity

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

for arbitrary α and β in $\mathbb{R}$.

For $m = 2$, we have

$$(\cos\theta)^2 = \frac{\cos(2\theta)+1}{2}. \tag{30}$$

Next, with $m = 3$,

$$\begin{aligned}
(\cos\theta)^3 &= \frac{\cos(2\theta)+1}{2}\cdot\cos\theta \\
&= \frac{\cos(2\theta)\cos\theta + \cos\theta}{2} \\
&= \frac{\frac{\cos(3\theta)+\cos\theta}{2} + \cos\theta}{2} \\
&= \frac{\cos(3\theta) + 3\cos\theta}{4}
\end{aligned} \tag{31}$$

Building on the pattern emerging from these calculations we now set out to prove the following fact.

Lemma 0.1 *Given θ in $\mathbb{R}$, for each $m = 1, 2, \ldots$, there exist scalars $a_{m,0}, \ldots, a_{m,m}$, independent of θ, such that*

$$(\cos\theta)^m = \sum_{k=0}^{m} a_{m,k}\cos(k\theta). \tag{32}$$

Proof. The proof proceeds by induction. The conclusion (32) is true for $m = 1$ (with $a_{1,0} = 0$ and $a_{1,1} = 1$), for $m = 2$ (with $a_{2,0} = \frac{1}{2}$, $a_{2,1} = 0$ and $a_{2.2} = \frac{1}{2}$) and for $m = 3$ (with $a_{3,0} = 0$, $a_{3,1} = \frac{3}{2}$, $a_{3,2} = 0$ and $a_{3,3} = \frac{1}{4}$).

Now assume (32) to hold for some $m \geq 2$. We note that

$$
\begin{aligned}
(\cos\theta)^{m+1} &= (\cos\theta)^m \cdot \cos\theta \\
&= \left(\sum_{k=0}^{m} a_{m,k}\cos(k\theta)\right)\cdot\cos\theta \\
&= a_{m,0}\cos\theta + \sum_{k=1}^{m} a_{m,k}\cos(k\theta)\cos\theta \\
&= a_{m,0}\cos\theta + \sum_{k=1}^{m} a_{m,k}\frac{\cos((k+1)\theta)+\cos((k-1)\theta)}{2} \\
&= a_{m,0}\cos\theta + \frac{1}{2}\sum_{k=1}^{m} a_{m,k}\cos((k+1)\theta) + \frac{1}{2}\sum_{k=1}^{m} a_{m,k}\cos((k-1)\theta) \\
&= a_{m,0}\cos\theta + \frac{1}{2}\sum_{k=2}^{m+1} a_{m,k-1}\cos(k\theta) + \frac{1}{2}\sum_{k=0}^{m-1} a_{m,k+1}\cos(k\theta) \\
&= \sum_{k=0}^{m+1} a_{m+1,k}\cos(k\theta) \qquad (33)
\end{aligned}
$$

with

$$
a_{m+1,k} = \begin{cases}
\frac{a_{m,1}}{2} & \text{if } k=0 \\
a_{m,0} + \frac{a_{m,2}}{2} & \text{if } k=1 \\
\frac{1}{2}\left(a_{m,k-1}+a_{m,k+1}\right) & \text{if } k=2,\ldots,m-1 \\
\frac{a_{m,m-1}}{2} & \text{if } k=m \\
\frac{a_{m,m}}{2} & \text{if } k=m+1
\end{cases}
$$

by direct inspection. This completes the proof of Lemma 0.1. ■