ENEE 420
FALL 2012
COMMUNICATIONS SYSTEMS

FOURIER ANALYSIS:

Consider a function a function $g : [0,T] \to \mathbb{C}$ such that

$$\int_0^T |g(t)|dt < \infty. \quad (1)$$

The fundamental frequency $f^\star$ associated with g is given by

$$f^\star := \frac{1}{T}.$$

Fourier coefficients

For any function $g : [0,T] \to \mathbb{C}$ satisfying the integrability condition (1), we write

$$c_n := \frac{1}{T}\int_0^T g(t)e^{-j2\pi f^\star nt}dt, \quad n = 0, \pm 1, \pm 2, \ldots$$

The quantity c_n is alway well defined under (1) and is called the Fourier coefficient of order n associated with g.

Fourier series

For any function $g : [0,T] \to \mathbb{C}$ satisfying the integrability condition (1), we introduce the *formal* series

$$\sum\nolimits_n c_n e^{j2\pi f^\star nt}, \quad t \in \mathbb{R}. \quad (2)$$

This series is known as the Fourier series associated with g. At this point it is not clear whether this series converges, and if it does, in what sense does the convergence take palce. These are tricky questions which we will not address here. However, it will be appropriate to think of (2 as a *representation* of g. In particular, it is appropriate to think of the collection of Fourier coefficients $\{c_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ as representing g!

Extending a function defined on a finite interval

Consider a function $h : [a,b] \to \mathbb{C}$ defined on some *finite* interval $[a,b]$. We can extend it into a function $\mathbb{R} \rightarrow \mathbb{C}$ (defined on the entire real line) in two non-equivalent ways.

First we can extend $h : [a,b] \rightarrow \mathbb{C}$ into the function $h_{\rm Ext} : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$h_{\rm Ext}(t) = \begin{cases} h(t) & \text{if } t \in [a,b] \\ \\ 0 & \text{if } t \notin [a,b] \end{cases} \tag{3}$$

Note that $h_{\rm Ext} : \mathbb{R} \rightarrow \mathbb{C}$ is a time-limited signal.

The second method for extending the definition of h to the entire real line is to use $h : [a,b] \rightarrow \mathbb{C}$ as the building block for a periodic function $h_{\rm Per} : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$\begin{aligned} & h_{\rm Per}(t) \\ = \ & h(t-(a+k(b-a))), \qquad \begin{array}{c} t \in [a+k(b-a), a+(k+1)(b-a)] \\ k = 0, \pm 1, \ldots \end{array} \end{aligned} \tag{4}$$

This definition is well posed only if the boundary condition

$$h(b) = h(a) \tag{5}$$

holds as this ensures

$$h_{\rm Per}(t) = \sum\nolimits_k h_{\rm Ext}(t-(a+k(b-a))), \quad t \in \mathbb{R}.$$

When (5) fails, it is customary to replace the boundary values by their average,[1] namely

$$h(a), h(b) \leftarrow \frac{1}{2}\left(h(a)+h(b)\right).$$

Assume the integrability condition

$$\int_a^b |h(t)| dt < \infty.$$

In that case, the Fourier series representation of the function $h : [a,b] \rightarrow \mathbb{C}$ is given by

$$\sum\nolimits_n c_{h,n} e^{j2\pi \frac{n}{b-a} t}, \quad t \in \mathbb{R} \tag{6}$$

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[1] This choice is dictated by results in Fourier analysis. Note that modifying h in a finite number of points will not change the Fourier coefficients.

where

$$c_{h,n} = \frac{1}{b-a}\int_a^b h(t)e^{-j2\pi\frac{n}{b-a}t}dt, \quad n = 0, \pm 1, \ldots 2, \ldots$$

Although Fourier series are associated (originally) with functions defined on finite intervals, it is customary to refer to (7) as the Fourier series of the function $h_{\mathrm{Per}} : \mathbb{R} \to \mathbb{C}$.

In a similar vein, consider a *periodic* function $g : \mathbb{R} \to \mathbb{C}$ of period T. If the integrability condition

$$\int_a^b |g(t)|dt < \infty$$

holds where $b = a + T$, then it is customary to say that the periodic signal $g : \mathbb{R} \to \mathbb{C}$ admits the Fourier series representation

$$\sum\nolimits_n c_{g,n}e^{j2\pi\frac{n}{T}t}, \quad t \in \mathbb{R} \tag{7}$$

where

$$c_{g,n} = \frac{1}{T}\int_a^b g(t)e^{-j2\pi\frac{n}{T}t}dt, \quad n = 0, \pm 1, \pm 2, \ldots$$

Square-integrable functions

A function $g : [0, T] \to \mathbb{C}$ is said to be *square-integrable* if

$$\int_0^T |g(t)|^2 dt < \infty \tag{8}$$

Note that if g satisfies (8) then it is necessarily integrable in the sense of (1). This is due to the fact that

$$|x| \leq 1 + |x|^2, \quad x \in \mathbb{R},$$

so that

$$\int_0^T |g(t)|dt \leq \int_0^T (1 + |g(t)|^2)dt = T + \int_0^T |g(t)|^2 dt.$$

Therefeore, $\int_0^T |g(t)|^2 dt < \infty$ implies $\int_0^T |g(t)|dt < \infty$.

Parseval's Theorem

For any pair of square-integrable functions $h, g : [0, T] \to \mathbb{C}$, we have

$$\int_0^T h(t)g(t)^\star dt = T\sum\nolimits_k c_{h,k}c_{g,k}^\star, \tag{9}$$

whence

$$\frac{1}{T}\int_0^T |g(t)|^2 dt = \sum\nolimits_k |c_{h,k}|^2 \tag{10}$$

with $\{c_{h,k},\ k = 0, \pm 1, \ldots\}$ and $\{c_{g,k},\ k = 0, \pm 1, \ldots\}$ denoting the Fourier coefficients of h and g, respectively.

We begin by noting that

$$\frac{1}{T}\int_0^T \left(e^{j2\pi k f_\star t}\right)\left(e^{j2\pi \ell f_\star t}\right)^\star dt = \delta(k,\ell), \quad k,\ell = 0, \pm 1, \ldots .$$

Consider two mapping $g, h : [0,T] \to \mathbb{C}$. Therefore, for any positive integer K, we have

$$\begin{aligned}
& \int_0^T \left(\sum_{k=-K}^{K} c_{h,k} e^{j2\pi k f_\star t}\right)\left(\sum_{\ell=-K}^{K} c_{g,\ell} e^{j2\pi \ell f_\star t}\right)^\star dt \\
= \;& \int_0^T \sum_{k=-K}^{K}\sum_{\ell=-K}^{K} \left(c_{h,k} e^{j2\pi k f_\star t}\right)\left(c_{g,\ell} e^{j2\pi \ell f_\star t}\right)^\star dt \\
= \;& \sum_{k=-K}^{K}\sum_{\ell=-K}^{K} \int_0^T \left(c_{h,k} e^{j2\pi k f_\star t}\right)\left(c_{g,\ell} e^{j2\pi \ell f_\star t}\right)^\star dt \\
= \;& \sum_{k=-K}^{K}\sum_{\ell=-K}^{K} c_{h,k} c_{g,\ell}^\star \int_0^T \left(e^{j2\pi k f_\star t}\right)\left(e^{j2\pi \ell f_\star t}\right)^\star dt \\
= \;& \sum_{k=-K}^{K}\sum_{\ell=-K}^{K} c_{h,k} c_{g,\ell}^\star \int_0^T e^{j2\pi k f_\star t} e^{-j2\pi \ell f_\star t} dt \\
= \;& \sum_{k=-K}^{K}\sum_{\ell=-K}^{K} c_{h,k} c_{g,\ell}^\star T \delta(k,\ell) \\
= \;& T \sum_{k=-K}^{K} c_{h,k} c_{g,k}^\star . \qquad (11)
\end{aligned}$$

Letting K go to infinity we see that

$$\begin{aligned}
& \int_0^T \left(\sum\nolimits_k c_{h,k} e^{j2\pi k f_\star t}\right)\left(\sum\nolimits_\ell c_{g,\ell} e^{j2\pi \ell f_\star t}\right)^\star dt \\
= \;& T \sum\nolimits_k c_{h,k} c_{g,k}^\star \qquad (12)
\end{aligned}$$

In particular,

$$\int_0^T h(t)g(t)^\star dt = T\sum\nolimits_k c_{h,k}c_{g,k}^\star \tag{13}$$

Therefore,

$$\frac{1}{T}\int_0^T |g(t)|^2 dt = \sum\nolimits_k |c_{h,k}|^2$$

Fourier transforms

Consider an integrable function $g : \mathbb{R} \to \mathbb{C}$ in the sense that

$$\int_{\mathbb{R}} |g(t)| dt < \infty. \tag{14}$$

The function $G : \mathbb{R} \to \mathbb{C}$ is defined by

$$G(f) := \int_{\mathbb{R}} f(t) e^{-j2\pi f t} dt, \quad f \in \mathbb{R}. \tag{15}$$

This definition is well posed under the integrability condition (14). It is customary to refer to the function $G : \mathbb{R} \to \mathbb{C}$ as the *Fourier transform* of g.