ENEE 420
FALL 2012
COMMUNICATIONS SYSTEMS

ANSWER KEY TO TEST # 2:

1. ______________________________

1.a
1.b
1.c
1.d The signals $z_+, z_- : \mathbb{R} \to \mathbb{R}$ implement USB and LSB modulation, respectively.

2. ______________________________

2.a The function g being defined on the finite interval $[-\frac{1}{2a}, \frac{1}{2a}]$, it is (square-)integrable since

$$|g(t)| \leq 1, \quad |t| \leq \frac{1}{2a}.$$

As a result, its Fourier coefficients are well defined and easily computed as follows: The length of the interval is a^{-1}, so that the fundamental frequency f_a appearing in the Fourier series expansion is given by $f_a = a$.

For each $k = 0, \pm 1, \pm 2, \ldots$, we have

$$\begin{aligned}
c_{g,k} &= \frac{1}{a^{-1}} \int_{-\frac{1}{4a}}^{\frac{1}{4a}} \sin(2\pi a t) e^{-j2\pi k f_a t} dt \\
&= \frac{a}{2j} \int_{-\frac{1}{4a}}^{\frac{1}{4a}} \left(e^{j2\pi a t} - e^{-j2\pi a t} \right) e^{-j2\pi k a t} dt \\
&= \frac{a}{2j} \int_{-\frac{1}{4a}}^{\frac{1}{4a}} \left(e^{-j2\pi a(k-1)t} - e^{-j2\pi a(k+1)t} \right) dt \\
&= \frac{1}{2j} \int_{-\frac{1}{4}}^{\frac{1}{4}} \left(e^{-j2\pi(k-1)s} - e^{-j2\pi(k+1)s} \right) ds \quad [s = at]
\end{aligned}$$

and direct inspection yields

$$c_{g,-k} = -c_{g,k}, \quad k = 1, 2, \ldots$$

Furthermore,

$$c_{g,0} = \frac{1}{a^{-1}} \int_{-\frac{1}{4a}}^{\frac{1}{4a}} \sin(2\pi a t) dt = 0 \tag{1.1}$$

by odd symmetry!

For $k \neq \pm 1$, straightforward calculations show that

$$\begin{aligned} c_{g,k} &= \frac{1}{2j} \left[\frac{e^{-j2\pi(k-1)s}}{-j2\pi(k-1)} - \frac{e^{-j2\pi(k+1)s}}{-j2\pi(k+1)} \right]_{-\frac{1}{4}}^{\frac{1}{4}} \\ &= \frac{1}{2j} \left[\frac{e^{-j(k-1)\frac{\pi}{2}} - e^{j(k-1)\frac{\pi}{2}}}{-j2\pi(k-1)} - \frac{e^{-j(k+1)\frac{\pi}{2}} - e^{j(k+1)\frac{\pi}{2}}}{-j2\pi(k+1)} \right] \\ &= \frac{1}{2j} \left[\frac{\sin\left((k-1)\frac{\pi}{2}\right)}{\pi(k-1)} - \frac{\sin\left((k+1)\frac{\pi}{2}\right)}{\pi(k+1)} \right]. \end{aligned} \tag{1.2}$$

Thus, for $k = 2\ell + 1$ with $\ell = 1, 2, \ldots$,

$$\sin\left((k-1)\frac{\pi}{2}\right) = \sin(\ell\pi) = 0 \quad \text{and} \quad \sin\left((k+1)\frac{\pi}{2}\right) = \sin((\ell+1)\pi) = 0,$$

whence

$$c_{g,2\ell+1} = 0, \quad \ell = 1, 2, \ldots \tag{1.3}$$

For $k = 2\ell$ with $\ell = 1, 2, \ldots$,

$$\begin{aligned} \sin\left((k \pm 1)\frac{\pi}{2}\right) &= \sin\left((2\ell \pm 1)\frac{\pi}{2}\right) \\ &= \sin\left(\ell\pi \pm \frac{\pi}{2}\right) \\ &= \sin(\ell\pi)\cos\left(\frac{\pi}{2}\right) \pm \cos(\ell\pi)\sin\left(\frac{\pi}{2}\right) \\ &= \pm\cos(\ell\pi) \\ &= \pm(-1)^{\ell}, \end{aligned} \tag{1.4}$$

whence

$$\begin{aligned} c_{g,2\ell} &= \frac{1}{2j} \left[\frac{-(-1)^{\ell}}{\pi(2\ell-1)} - \frac{(-1)^{\ell}}{\pi(2\ell+1)} \right] \\ &= \frac{1}{2j} \cdot \frac{-4\ell(-1)^{\ell}}{\pi(4\ell^2-1)} \\ &= \frac{1}{j} \cdot \frac{-2\ell(-1)^{\ell}}{\pi(4\ell^2-1)}, \quad \ell = 1, 2, \ldots \end{aligned} \tag{1.5}$$

Finally, for $k = 1$,

$$\begin{aligned} c_{g,1} &= \frac{1}{2j}\int_{-\frac{1}{4}}^{\frac{1}{4}} \left(1 - e^{-j4\pi s}\right) ds \\ &= \frac{1}{2j}\left[\frac{1}{2} - \left[\frac{e^{-j4\pi s}}{-j4\pi}\right]_{-\frac{1}{4}}^{\frac{1}{4}}\right] \\ &= \frac{1}{2j}\left[\frac{1}{2} - \frac{e^{-j\pi} - e^{j\pi}}{-j4\pi}\right] \\ &= \frac{1}{2j}\left[\frac{1}{2} - \frac{\sin(\pi)}{2\pi}\right] = \frac{1}{4j}. \end{aligned} \tag{1.6}$$

The Fourier series expansion of g is given by

$$\begin{aligned} &\sum_k c_{g,k} e^{j2\pi k f_a t} \\ &= c_{g,0} + \sum_{k=1}^{\infty} c_{g,k}\left(e^{j2\pi kat} - e^{-j2\pi kat}\right) \\ &= 2j \cdot \sum_{k=1}^{\infty} c_{g,k}\sin(2\pi kat) \\ &= 2j \cdot \left(c_{g,1}\sin(2\pi at) + \sum_{\ell=1}^{\infty} c_{g,2\ell}\sin(4\pi\ell at)\right) \\ &= \frac{1}{2}\sin(2\pi at) + 2j \cdot \sum_{\ell=1}^{\infty} c_{g,2\ell}\sin(4\pi\ell at) \\ &= \frac{1}{2}\sin(2\pi at) - \frac{4}{\pi}\sum_{\ell=1}^{\infty}\frac{(-1)^{\ell}\ell}{4\ell^2 - 1}\sin(4\pi\ell at), \quad t \in \mathbb{R}. \end{aligned} \tag{1.7}$$

2.b The function h being defined on the finite interval $[-\frac{1}{2a}, \frac{1}{2a}]$, it is (square-)integrable since

$$|h(t)| \le 1, \quad |t| \le \frac{1}{4a}.$$

This time, the length of the interval is $(2a)^{-1}$, so that the fundamental frequency appearing in the Fourier series expansion for h is given by $2f_a = 2a$. In fact, we have

$$c_{h,k} = 2a\int_{-\frac{1}{4a}}^{\frac{1}{4a}} \sin(2\pi at)e^{-j2\pi k(2a)t}dt = 2c_{g,2k}, \quad k = 0, \pm1, \pm2, \ldots$$

so it is still the case that

$$c_{h,0} = 0, \quad c_{h,-k} = -c_{h,k}, \quad k = 0, \pm1, \pm2, \ldots$$

However, the Fourier series expansion of h is now given by

$$
\begin{aligned}
&\sum_k c_{h,k} e^{j2\pi k 2 f_a t} \\
&= c_{h,0} + \sum_{k=1}^{\infty} c_{h,k} \left(e^{j4\pi kat} - e^{-j4\pi kat} \right) \\
&= 4j \cdot \sum_{k=1}^{\infty} c_{g,2k} \sin\left(4\pi kat\right) \\
&= 4j \cdot \left(\sum_{k=1}^{\infty} \frac{1}{j} \cdot \frac{-2k(-1)^k}{\pi(4k^2-1)} \cdot \sin\left(4\pi kat\right) \right) \\
&= -\frac{8}{\pi} \sum_{k=1}^{\infty} \frac{k(-1)^k}{4k^2-1} \cdot \sin\left(4\pi kat\right), \quad t \in \mathbb{R}.
\end{aligned}
\tag{1.8}
$$

It should be clear from these calculations that the Fourier series expansions of the functions g and h are indeed different as they should be since they are *different* functions!

3.

3.a We have

$$
H_{\rm VSB}(f) = \begin{cases}
0 & \text{if } 0 \le f \le f_c - \frac{B}{2} \\
\frac{1}{B}\left(f - f_c + \frac{B}{2}\right) & \text{if } f_c - \frac{B}{2} \le f \le f_c + \frac{B}{2} \\
1 & \text{if } f_c + \frac{B}{2} \le f \le f_c + B \\
0 & \text{if } f_c + B < f.
\end{cases}
$$

with

$$
H_{\rm VSB}(-f) = H_{\rm VSB}(f), \quad f \ge 0.
$$

From this expression for $H_{\rm VSB}$ (or graphically), it is easy to check that

$$
H_{\rm VSB}(f - f_c) + H_{\rm VSB}(f + f_c) = \begin{cases}
1 & \text{if } |f| \le B \\
0 & \text{if } |f| > B
\end{cases}
\tag{1.9}
$$

and the requisite condition holds.

3.b The transmission bandwidth B_T is given by

$$
B_T = B + \frac{B}{2} = \frac{3B}{2}.
$$

3.c By construction

$$
s_{\rm VSB} = h_{\rm VSB} \star y_{DSB-SC}
$$

so that

$$S_{\rm VSB}(f) = H_{\rm VSB}(f) S_{DSB-SC}(f), \quad f \in \mathbb{R}$$

with

$$S_{DSB-SC}(f) = \frac{A_c}{2}\left(M(f-f_c) + M(f+f_c)\right), \quad f \in \mathbb{R}.$$

Here,

$$M(f) = \frac{A_m}{2}\left(\delta(f-f_m) + \delta(f+f_m)\right), \quad f \in \mathbb{R}$$

so that

$$\begin{aligned} S_{DSB-SC}(f) &= \frac{A_c A_m}{4}\left(\delta(f-f_c-f_m) + \delta(f-f_c+f_m)\right) \\ &\quad + \frac{A_c A_m}{4}\left(\delta(f+f_c-f_m) + \delta(f+f_c+f_m)\right). \end{aligned} \tag{1.10}$$

It is now plain that

$$\begin{aligned} & S_{\rm VSB}(f) \\ &= H_{\rm VSB}(f) S_{DSB-SC}(f) \\ &= \frac{A_c A_m}{4}\left(H_{\rm VSB}(f_c+f_m)\delta(f-f_c-f_m) + H_{\rm VSB}(f_c-f_m)\delta(f-f_c+f_m)\right) \\ &\quad + \frac{A_c A_m}{4}\left(H_{\rm VSB}(-f_c+f_m)\delta(f+f_c-f_m) + H_{\rm VSB}(-f_c-f_m)\delta(f+f_c+f_m)\right). \end{aligned}$$

When $\frac{1}{2} < c < 1$, we note that

$$H_{\rm VSB}(f_c+f_m) = H_{\rm VSB}(-(f_c+f_m)) = 1$$

and

$$H_{\rm VSB}(f_c-f_m) = H_{\rm VSB}(-(f_c-f_m)) = 0.$$

Therefore, we conclude that

$$S_{\rm VSB}(f) = \frac{A_c A_m}{4}\left(\delta(f-(f_c+f_m)) + \delta(f+(f_c+f_m))\right), \quad f \in \mathbb{R},$$

whence

$$s_{\rm VSB}(t) = \frac{A_c A_m}{2}\cos\left(2\pi(f_c+f_m)t\right), \quad t \in \mathbb{R}.$$

3.d When $0 < c < \frac{1}{2}$, we now have

$$H_{\rm VSB}(f_c+f_m) = H_{\rm VSB}(-(f_c+f_m)) = \frac{1}{B}\left(f_m + \frac{B}{2}\right) = c + \frac{1}{2}$$

and

$$H_{\rm VSB}(f_c-f_m) = H_{\rm VSB}(-(f_c-f_m)) = \frac{1}{B}\left(-f_m + \frac{B}{2}\right) = -c + \frac{1}{2}$$

whence

$$\begin{aligned} & S_{\mathrm{VSB}}(f) \\ = & \frac{A_c A_m}{4}\left(\left(c+\frac{1}{2}\right)\delta(f-f_c-f_m)+\left(-c+\frac{1}{2}\right)\delta(f-f_c+f_m)\right) \\ & + \frac{A_c A_m}{4}\left(\left(-c+\frac{1}{2}\right)\delta(f+f_c-f_m)+\left(c+\frac{1}{2}\right)\delta(f+f_c+f_m)\right). \end{aligned}$$

It is now easy to see that

$$s_{\mathrm{VSB}}(t)=\frac{A_c A_m}{2}\left(\left(c+\frac{1}{2}\right)\cos\left(2\pi(f_c+f_m)t\right)+\left(-c+\frac{1}{2}\right)\cos\left(2\pi(f_c-f_m)t\right)\right)$$

for each t in $\mathbb{R}$.

Note that

$$\cos\left(2\pi(f_c\pm f_m)t\right)=\cos\left(2\pi f_c t\right)\cos\left(2\pi f_m t\right)\mp\sin\left(2\pi f_c t\right)\sin\left(2\pi f_m t\right)$$

so that

$$\begin{aligned} & \left(c+\frac{1}{2}\right)\cos\left(2\pi(f_c+f_m)t\right)+\left(-c+\frac{1}{2}\right)\cos\left(2\pi(f_c-f_m)t\right) \\ = & \left(c+\frac{1}{2}\right)\left[\cos\left(2\pi f_c t\right)\cos\left(2\pi f_m t\right)-\sin\left(2\pi f_c t\right)\sin\left(2\pi f_m t\right)\right] \\ & + \left(-c+\frac{1}{2}\right)\left[\cos\left(2\pi f_c t\right)\cos\left(2\pi f_m t\right)+\sin\left(2\pi f_c t\right)\sin\left(2\pi f_m t\right)\right] \\ = & \cos\left(2\pi f_c t\right)\cos\left(2\pi f_m t\right)-2c\sin\left(2\pi f_c t\right)\sin\left(2\pi f_m t\right) \qquad (1.11) \end{aligned}$$

Therefore,

$$\begin{aligned} s_{\mathrm{VSB}}(t) &= \frac{A_c A_m}{2}\left(\cos\left(2\pi f_c t\right)\cos\left(2\pi f_m t\right)-2c\sin\left(2\pi f_c t\right)\sin\left(2\pi f_m t\right)\right) \\ &= A_c\left(m_1(t)\cos\left(2\pi f_c t\right)+m_2(t)\sin\left(2\pi f_c t\right)\right) \qquad (1.12) \end{aligned}$$

where

$$m_1(t)=\frac{m(t)}{2} \quad \text{and} \quad m_2(t)=-cA_m\sin\left(2\pi f_m t\right), \quad t\in\mathbb{R}.$$

4. ____

4.a The signal $t\rightarrow\varepsilon(t)$ is periodic with period T, and we have

$$\int_{-\frac{T}{2}}^{\frac{T}{2}}\left|e^{jk_P m(t)}\right|^2 dt=\int_{-\frac{T}{2}}^{\frac{T}{2}} dt=T.$$

Thus, the signal $t\rightarrow\varepsilon(t)$ admits a Fourier series representation of the form

$$\varepsilon(t)=\sum\nolimits_k c_k e^{j2\pi\frac{k}{T}t}, \quad t\in\mathbb{R}$$

with Fourier coefficients

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{jk_P m(t)} e^{-j2\pi\frac{k}{T}t}dt, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Now, for a given $k = 0, \pm 1, \pm 2, \ldots$, we get

$$\begin{aligned} c_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{jk_P m(t)} e^{-j2\pi\frac{k}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{2}} e^{jk_P m(t)} e^{-j2\pi\frac{k}{T}t}dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{0} e^{-jk_P} e^{-j2\pi\frac{k}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{2}} e^{jk_P} e^{-j2\pi\frac{k}{T}t}dt \\ &= \frac{1}{T}\int_{0}^{\frac{T}{2}} e^{-jk_P} e^{j2\pi\frac{k}{T}t}dt + \frac{1}{T}\int_{0}^{\frac{T}{2}} e^{jk_P} e^{-j2\pi\frac{k}{T}t}dt. \end{aligned} \tag{1.13}$$

For $k = 0$,

$$c_0 = \frac{1}{2}\left(e^{jk_P} + e^{-jk_P}\right) = \cos\left(k_P\right).$$

For $k = \pm 1, \pm 2, \ldots$,

$$\begin{aligned} c_k &= \frac{e^{-jk_P}}{T}\left(\frac{e^{j\pi k} - 1}{j2\pi\frac{k}{T}}\right) + \frac{e^{jk_P}}{T}\left(\frac{e^{-j\pi k} - 1}{-j2\pi\frac{k}{T}}\right) \\ &= e^{-jk_P}\left(\frac{e^{j\pi k} - 1}{j2\pi k}\right) + e^{jk_P}\left(\frac{e^{-j\pi k} - 1}{-j2\pi k}\right) \\ &= e^{-jk_P}\left(\frac{(-1)^k - 1}{j2\pi k}\right) + e^{jk_P}\left(\frac{(-1)^k - 1}{-j2\pi k}\right) \\ &= \left(\frac{(-1)^k - 1}{\pi k}\right)\cdot\left(\frac{e^{-jk_P} - e^{jk_P}}{2j}\right) \\ &= -\left(\frac{(-1)^k - 1}{\pi k}\right)\sin\left(k_P\right). \end{aligned} \tag{1.14}$$

As a result,

$$\begin{aligned} \varepsilon(t) &= \cos\left(k_P\right) - \sin\left(k_P\right)\sum\nolimits_{k\neq 0}\left(\frac{(-1)^k - 1}{\pi k}\right)e^{j2\pi\frac{k}{T}t} \\ &= \cos\left(k_P\right) - \sin\left(k_P\right)\sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{\pi k}\right)\left(e^{j2\pi\frac{k}{T}t} - e^{-j2\pi\frac{k}{T}t}\right) \\ &= \cos\left(k_P\right) - 2j\sin\left(k_P\right)\sum_{k=1}^{\infty}\left(\frac{(-1)^k - 1}{\pi k}\right)\sin\left(2\pi\frac{k}{T}t\right) \\ &= \cos\left(k_P\right) + j\frac{4\sin\left(k_P\right)}{\pi}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\sin\left(2\pi\frac{2\ell+1}{T}t\right), \quad t \in \mathbb{R}. \end{aligned}$$

4.b The PM signal $s_{\rm PM} : \mathbb{R} \to \mathbb{R}$ can be evaluated as follows: For each t in $\mathbb{R}$, we get

$$\begin{aligned}
s_{\rm PM}(t) &= A_c \cos\left(2\pi f_c t + k_P m(t)\right) \\
&= A_c {\rm Re}\left(e^{j(2\pi f_c t + k_P m(t))}\right) \\
&= A_c {\rm Re}\left(e^{j2\pi f_c t}\varepsilon(t)\right) \\
&= A_c {\rm Re}\left(e^{j2\pi f_c t}\left(\cos\left(k_P\right) + j\frac{4\sin\left(k_P\right)}{\pi}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\sin\left(2\pi\frac{2\ell+1}{T}t\right)\right)\right) \\
&= A_c \cos\left(k_P\right)\cos\left(2\pi f_c t\right) \\
&\quad - \frac{4A_c}{\pi}\sin\left(k_P\right)\sin\left(2\pi f_c t\right)\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\sin\left(2\pi\frac{2\ell+1}{T}t\right) \\
&= A_c \cos\left(k_P\right)\cos\left(2\pi f_c t\right) \\
&\quad - \frac{2A_c}{\pi}\sin\left(k_P\right)\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\cos\left(2\pi\left(f_c - \frac{2\ell+1}{T}\right)t\right) \\
&\quad + \frac{2A_c}{\pi}\sin\left(k_P\right)\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\cos\left(2\pi\left(f_c + \frac{2\ell+1}{T}\right)t\right).
\end{aligned} \tag{1.15}$$

4.c The PM signal $s_{\rm PM} : \mathbb{R} \to \mathbb{R}$ has frequency content at the discrete frequencies

$$f = \pm f_c \quad \text{and} \quad f = \pm\left(f_c \pm \frac{2\ell+1}{T}\right), \quad \ell = 0, 1, \ldots$$

The power of the PM signal $s_{\rm PM} : \mathbb{R} \to \mathbb{R}$ can now be computed in the usual manner:

$$\begin{aligned}
\mathtt{P}_{s_{\rm PM}} &= \lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}\left|s_{\rm PM}(t)\right|^2 dt \\
&= A_c^2 \cdot \lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}\left|\cos\left(2\pi f_c t + k_P m(t)\right)\right|^2 dt \\
&= \frac{A_c^2}{2}\left(1 + \lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}\cos\left(4\pi f_c t + 2k_P m(t)\right) dt\right) \\
&= \frac{A_c^2}{2}
\end{aligned} \tag{1.16}$$

because

$$\lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}\cos\left(4\pi f_c t + 2k_P m(t)\right) dt = 0 \tag{1.17}$$

upon using the form of m.
