Lecture 1 Continued

The physical example of the inverted pendulum with friction and external forcing determined by proportional feedback shows that feedback can produce a key benefit:

overcome instability

More broadly, feedback captures the principle that,

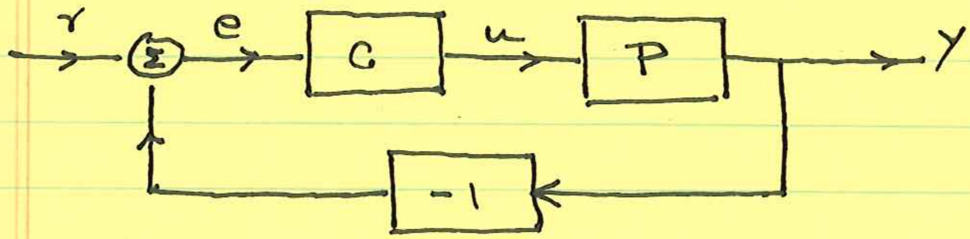
> in taking action to achieve a purpose one should observe whether a purpose is being met, and if not, revise one's action based on observation.

This principle is illustrated in nature across scales, and exploited in technology.

Benefits of feedback – responsiveness to measurements, robustness to modeling error, reshaping of dynamics, modularity through nested loops, ...

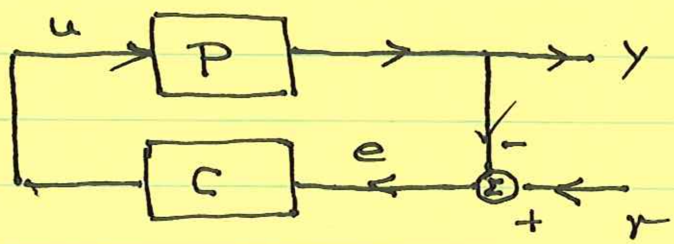
Drawbacks of feedback – noise in measurements enters feedback loop, instability through positive feedback, complexity, ...

We represent systems (physical plants) with feedback controllers coupled to them using diagrams of signal flow, such as



$u :=$ input
$y :=$ output
$r :=$ reference
$e :=$ error

or equivalently,



Here P denotes plant, C the controller, and the diagrams capture the equations

$$y = Pu \qquad (1)$$

error: $$e = r - y \qquad (2)$$

$$u = Ce \qquad (3)$$

What is the interpretation of (1) and (3) in mathematical terms? What is the meaning of reference r?

We start by noting that a classic, useful, effective controller is

$$u = k_p e + k_i \int_0^t e(\tau)d\tau + k_d \frac{de}{dt} \qquad (4)$$

proportional term

integral term

derivative term

For suitable choices k_p, k_i, k_d, this <u>linear</u> controller, known as PID controller, reaps some of the benefits of feedback. The proportional term captures control based on present. The derivative term gives weight to the future (or predicted future) while the integral term draws on past error (or error history).

We have thus illustrated what equation (3) might look like.

Observe that (4) can be re-written as

$$\frac{du}{dt} = k_p \frac{de}{dt} + k_i e + k_d \frac{d^2e}{dt^2} \quad (5)$$

or succintly

$$Du = (k_d D^2 + k_p D + k_i)e \quad (6)$$

where $D = \frac{d}{dt}$ denotes the operation of differentiation. More generally the controller might take the form

$$\boxed{a_c(D)\, u = b_c(D)\, e} \quad (7)$$

where

$$a_c(D) = D^m + a_{c1} D^{m-1} + \cdots + a_{c(m-1)} D + a_{cm}$$

and

$$b_c(D) = b_{c1} D^{r-1} + b_{c2} D^{r-2} + \cdots + b_{cr}$$

with coefficients real, and degrees m and r. In a PID controller (6), $m=1$ and $r=3$.

· Linear controllers of the form (7) go together with, and are effective for linear plants $\underline{P}$ governed by

$$\boxed{a_{\underline{P}}(D)\,y = b_{\underline{P}}(D)\,u} \qquad (8)$$

where

$$a_{\underline{P}}(D) = D^n + a_{\underline{P}1}D^{n-1} + \dots + a_{\underline{P}(n-1)}D + a_{\underline{P}n}$$

and

$$b_{\underline{P}}(D) = b_{P1}D^{n-1} + b_{\underline{P}2}D^{n-2} + \dots + b_{P(n-1)}D + b_{\underline{P}n}$$

are polynomial differential operators.

For the linearized inverted pendulum with control (i.e. equation 5′ in Lecture 1 with f on the right hand side), dropping the subscript $\underline{P}$, the plant equation is

$$a(D)\,\delta = f$$

i.e. $$a(D) = D^2 + bD - \frac{g}{l}$$

and

$$b(D) = 1$$

We can now be more precise what equations (1) and (3) mean. They represent equations (8) for the plant and (7) for the controller, respectively. These are of course linear differential equations and hence we expect their solutions to be made up of two terms, one depending on ~~on~~ initial conditions and one not. Specifically, for

$$a(D)y = b(D)u$$

general solution, $y = y_h + y_p$ (SPLITTING)
where the homogeneous solution y_h satisfies, (y_h depends on initial conditions),

$$a(D)y_h = 0$$

and the particular ~~solution~~ solution y_p satisfies

$$a(D)y_p = b(D)u$$

Given a specific time function u, is the splitting above unique? This is best answered using the methods of linear algebra and state space representation.

The particular solution y_p can be <u>teased out</u> by the following:

Pick $s \in \mathbb{C}$ a complex number.
Define input
$$u(t) = e^{st}.$$

From $D^k u(t) = s^k e^{st}$ $k = 0, 1, 2, \ldots$
it follows that

$$y_p(t) = P(s)\, e^{st}$$

is a particular solution satisfying

$$a(D) y_p = b(D)\, e^{st}$$

provided

$$\boxed{P(s) = \frac{b(s)}{a(s)}}$$

Observe that we have now <u>conveniently</u> swapped s for D as argument for the polynomials $\underline{a}$ and $\underline{b}$. As s varies so does $P(s)$ and we call $P(s)$ the <u>transfer function</u> of the plant.