Lecture 2 ENEE 460 Control Systems

Here we simply capture some of the modeling assumptions used in discussing feedback principles in Chapter 2 of Astrom-Murray (2nd Edition). A plant P has inputs and outputs. Initially focus on scalar inputs and scalar outputs (as functions of time). The plant model is polynomial in the sense that output $y(t)$ is driven by the input $u(t)$ according to equation

$$\boxed{a(D)y = b(D)u}$$

where

$$a(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$$

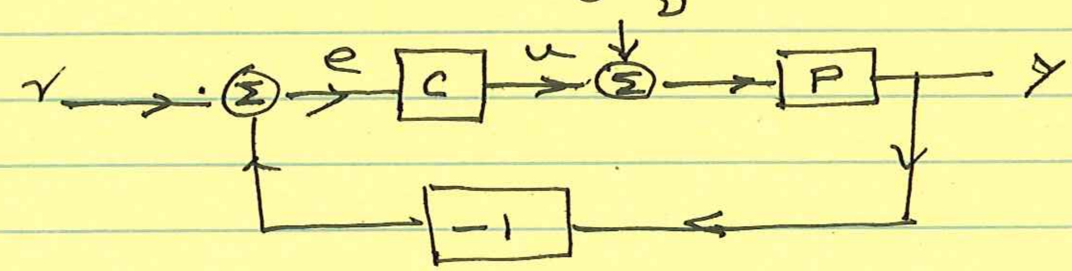
$$b(D) = b_1 D^{n-1} + b_2 D^{n-2} + \cdots + b_n$$

are polynomials in the symbol D which denotes the operation $\frac{d}{dt}$ of differentiation. By convention,

$$\left(\frac{d}{dt}\right)^0 y = D^0 y = y$$

and $$\left(\frac{d}{dt}\right)^{k+1} y = D^{k+1} y = D(D^k y).$$

Similar models apply to controllers as well — after all they too are implemented as physical plants. When we put together controllers and plants in a feedback configuration



The governing equations are:

$$a_p(D)y = b_p(D)(u+v) \quad (1)$$

$$e = r - y \quad (2)$$

$$a_c(D)u = b_c(D)e \quad (3)$$

Here the subscript(s) P (or C) refer to the plant (or controller).

Pre-multiplying (1) by a_c and using commutativity of multiplication of the polynomials, we get.

$$(a_c a_p + b_c b_p)y = b_c b_p r + a_c b_p v \quad (4)$$

Here we have suppressed the D symbol.

Pre-multiplying equation (3) by a_p and again exploiting commutativity, we get,

$$(a_c a_p + b_c b_p)\, u = \cancel{\quad} - b_c b_p \nu + b_c a_p r$$

We can associate the transfer functions equation

$$y = G_{yr}\, r + G_{y\nu}\, \nu$$

$$= \frac{PC}{1+PC} r + \frac{P}{1+PC} \nu$$

$$u = \frac{C}{1+PC} r \cancel{+} - \frac{CP}{1+PC} \nu$$

Some insights from examples

(1) $P = \frac{b}{s+a}$; $C = k_p$ (proportional control)

$$G_{y\nu} = \frac{b}{s + (a + bk_p)}$$

stability $\Leftrightarrow$ $a + bk_p > 0$

steady state error $\frac{b}{a+bk_p}$

High gain $\Rightarrow$ lower error

2 $$u = k_p e + k_i \int_0^t e(\tau)\,d\tau$$

$$Du = k_p De + k_i e$$

$$= (k_p D + k_i) e$$

$$\Rightarrow \quad C = (k_p D + k_i) / D$$

equivalently

$C(s) = (k_p s + k_i)/s$ is the

controller transfer function.

$$G_{yv} = \frac{P}{1 + PC}$$

$$= \frac{bs}{s^2 + s(a + bk_p) + bk_i}$$

stability $\Leftrightarrow$ $a + bk_p > 0$, $bk_i > 0$

In general we can write

$$s^2 + s(a + bk_p) + bk_i = s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2$$

~~for~~ <u>specified</u> $\sigma_d > 0$, $\omega_d \in \mathbb{R}$.

$$\Rightarrow \left.\begin{aligned} k_p &= \frac{2\sigma_d - a}{b} \\ k_i &= (\sigma_d^2 + \omega_d^2)/b \end{aligned}\right\} \rightarrow \text{control design}$$

One can also write

$$s^2 + 2\sigma_d s + \sigma_d^2 + \omega_d^2$$

$$= s^2 + 2\zeta_c \omega_c s + \omega_c^2 \Big\rangle \ \underline{\text{specification}}$$

where $\zeta_c = \dfrac{\sigma_d}{\sqrt{\sigma_d^2 + \omega_d^2}}$ lies in $(0, 1)$

$= \sigma_d / \omega_c$

for stability.

ζ_c is the damping factor

ω_c is the undamped natural frequency.

Design: $k_p = (2\sigma_d - a)/b = (2\omega_c \zeta_c - a)/b$

$k_i = (\sigma_d^2 + \omega_d^2)/b = \omega_c^2/b$

Specification of ω_c, ζ_c is based on the <u>step response</u> of

$$G_{yr} = \frac{PC}{1+PC}$$

$$= \frac{b(k_p s + k_i)}{s^2 + s(a + bk_p) + bk_i}$$

$$= \frac{(2\omega_c\zeta_c - a)s + \omega_c^2}{s^2 + 2\zeta_c\omega_c s + \omega_c^2} \quad \boxtimes$$

3. In (2) there are no constraints on how to specify ω_c, ζ_c. But if there are modeling errors, there arise constraints

If <u>true</u> $P(s) = \dfrac{b}{(s+a)(1+sT)}$

then the closed loop polynomial is

$$s(s+a)(1+sT) + k_p s + k_i$$

$$= s^3 + s^2(1+aT) + 2\zeta_c\omega_c s + \omega_c^2$$

$$\Rightarrow \quad \omega_c < 2\zeta_c(1 + aT)/T$$

for stability.

So step response cannot be arbitrarily fast.

<u>Readings to do</u>: Sections 2.1, 2.2, 2.3, 2.4 (skipping nonlinear aspects), 2.5 (integral action), ~~2.5~~ 2.6 (segway), 2.7.

Understand the step response simulation reported in Figures 2.4, 2.6