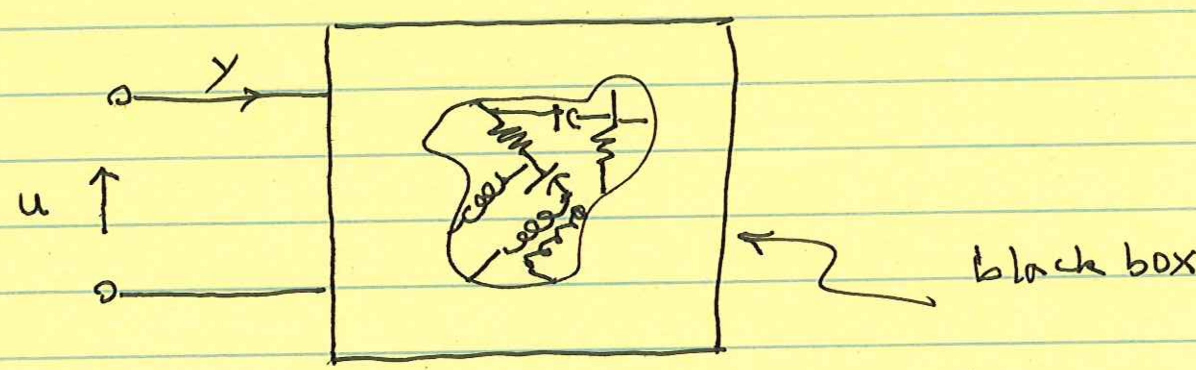
Lecture 3 ENEE 460 {Control Systems}

Think of a system made up of passive electrical elements (linear resistors, capacitors, inductors, ideal transformers and gyrators) forming a circuit accessible through a port (= pair of terminals):



Let u denote (driving) voltage and y denote output current. If you have the ability to look inside the black box and measure various voltages and currents (all respecting KCL and KVL) then you should be able to determine the relationship between u and y. Such internal information can be called hidden / latent information or more commonly state information.

by solving circuit equations

~~It is the external behavior of the system that can~~

It is a <u>subtle idea</u> developed later that in fact one can find a single function $\xi(t)$ that contains <u>all</u> internal information. It satisfies

$$\boxed{a(D)\xi = u = \text{input}} \quad \underline{1}$$

and the output y is given by

$$\boxed{y = b(D)\xi}. \quad \underline{2}$$

Observe that, in this case,

$$\begin{aligned} a(D)y &= a(D)\,b(D)\xi \\ &= b(D)\,a(D)\xi \\ &= b(D)u, \end{aligned} \quad \underline{3}$$

So we get the type of <u>plant model</u> we have been discussing out of the equations 1 and 2 that incorporates the latent variable ξ. What is the value of 1 & 2 ? It is a <u>natural path</u> to writing down models for plants in terms of matrix-vector differential equations. Here are the basic steps:

Let $\quad x_1(t) = \xi(t) \qquad \underline{4}$

Similarly define

$$x_k(t) = D^{k-1}\xi(t) \qquad \underline{5}$$

$$k = 1, 2, \cdots, n$$

Assemble these components into a vector

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \\ \\ \\ x_n(t) \end{bmatrix} \qquad \underline{6}$$

Then $\dot{x}$ = vector of derivatives $\dot{x}_i(t)$, $i = 1, 2, \cdots, n$ obeys the following

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_k = x_{k+1} \qquad k = 1, 2, \ldots, n-1$$

and

$$\dot{x}_n = D^n \xi$$

$$= -a_1 D^{n-1}\xi - a_2 D^{n-2}\xi$$
$$\cdots - a_n \xi + u$$

(from $\underline{1}$)

$$= -a_n x_1 - a_{n-1} x_2$$
$$\cdots - a_1 x_n + u$$

These scalar component equations can be collected as

$$\boxed{\dot{x} = Ax + e_n u} \qquad \underline{7}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & & \cdots & -a_1 \end{bmatrix} \qquad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \underline{8}$$

Now, from $\underline{2}$

$$y = b_1 D^{n-1}\xi + b_2 D^{n-2}\xi + \cdots + b_n \xi$$

$$= b_n x_1 + b_{n-1} x_2 + \cdots + b_1 x_n$$

$$= (b_n, b_{n-1}, \cdots, b_1)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= c\, x$$

where the row vector

$$c \triangleq (b_n, b_{n-1}, \cdots, b_1).$$

We thus have what we call a state variable model (or state space model), or internal model

$$\boxed{\begin{aligned} \dot{x} &= A x + e_n u \\ y &= c\, x \end{aligned}} \qquad \underline{9}$$

which yields the same plant model

$$a(D)\, y = b(D)\, u \qquad (\text{from } \underline{3})$$

One of the first things we can do with this state space model is to obtain the <u>transfer</u> <u>function</u> using Laplace transforms.

Take the Laplace transform of both sides of the system of equations <u>9</u>, denoting by

$\underline{X}(s)$ the vector of $X_i(s)$, $i = 1, 2, \cdots, n$, where

$$X_i(s) = \int_0^\infty e^{-st} x_i(t)\,dt.$$

Recall that Laplace transform

$$\mathcal{L}(\dot{x}_i(t))$$

$$= s\underline{X}_i(s) - x_i(0).$$

(use integration by parts to derive this).

Then

$$s\underline{X}(s) - \mathbf{x}(0) = A\underline{X}(s) + e_n U(s)$$
$$Y(s) = C\underline{X}(s)$$

It follows that

$$(sI - A)X(s) = x(0) + e_n U(s)$$

$\Rightarrow$

$$X(s) = (sI-A)^{-1} x(0) + (sI-A)^{-1} e_n \, U(s)$$

well defined for all complex numbers s except the eigenvalues of A, where $\det(sI - A) = 0$

Thus,

$$Y(s) = c(sI-A)^{-1} x(0) + c(sI-A)^{-1} e_n \, U(s)$$

The expression

$$\boxed{c\,(sI - A)^{-1} e_n}$$

is just the <u>transfer function</u>

$$\boxed{G(s) = b(s)/a(s)}\,.$$

We now have a task of computing $G(s)$ using the above formula

Let

$$(sI - A)^{-1} e_n = \eta$$

where $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$.

Then

$$e_n = (sI - A)\eta$$

Equivalently,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 & & & 0 \\ 0 & s & -1 & & & \\ 0 & 0 & s & -1 & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ a_n & a_{n-1} & a_{n-2} & \cdots & & s+a_1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_n \end{bmatrix}$$

and hence

$$0 = s\eta_1 - \eta_2$$

$$0 = s\eta_2 - \eta_3$$

$$\vdots$$

$$0 = s\eta_{n-1} - \eta_n$$

$$1 = a_n\eta_1 + a_{n-1}\eta_2 + \cdots + a_2\eta_{n-1} + (s+a_1)\eta_n$$

$$\Leftrightarrow \quad \eta_2 = s\eta_1$$

$$\eta_3 = s^2\eta_1$$

$$\eta_4 = s^3\eta_1$$

$$\vdots$$

$$\eta_n = s^{n-1}\eta_1$$

$$1 = \left(a_n + a_{n-1}s + a_{n-2}s^2 + \cdots + a_1 s^{n-1} + s^n\right)\eta_1$$

$$\Longleftrightarrow \quad \eta_1 = \frac{1}{s^2 + a_1 s^{n-1} + \cdots + a_n} = \frac{1}{a(s)}$$

After back-substitution for η_k
$k = 1, 2, \ldots, n$ in the formula

$$c\,(sI - A)^{-1} e_n$$

$$= (b_n,\ b_{n-1},\ \ldots,\ b_1) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \\ \eta_n \end{pmatrix}$$

$$= (b_n,\ b_{n-1},\ \ldots,\ b_1) \begin{pmatrix} 1 \\ s \\ \\ s^{n-1} \end{pmatrix} \frac{1}{a(s)}$$

$$= \frac{b(s)}{a(s)}$$

as claimed earlier

We can now gather the results so far into a diagram as shown on the next page

$$\dot{x} = Ax + e_n u$$
$$y = Cx$$

State Space Model

$x_k = D^{k-1}\xi$

$k = 1, 2, \dots, n$

Laplace Transform

$$a(D)\xi = u$$
$$y = b(D)\xi$$
$$\Downarrow$$
$$a(D)y = b(D)u$$

Polynomial Operater
Time Domain Model

$$G(s) = \frac{b(s)}{a(s)}$$

Transfer Function

Computing particular solution for complex exponential inputs.