ENEE 620
FALL 2011
RANDOM PROCESSES
IN COMMUNICATION AND CONTROL

ANSWER KEY TO TEST # 1:

1. ______________________________

1.a. Fix $x > 0$ and $s > 0$. It is plain that

$$\begin{aligned}
\mathbb{P}[X \leq x, X+Y \leq s] &= \mathbb{E}[\mathbf{1}[X \leq x, X+Y \leq s]] \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}[X \leq x, X+Y \leq s] | X]] \\
&= \mathbb{E}[\mathbf{1}[X \leq x] \mathbb{E}[\mathbf{1}[X+Y \leq s] | X]] \\
&= \mathbb{E}[\mathbf{1}[X \leq x] \mathbb{E}[\mathbf{1}[\star + Y \leq s]]_{\star = X}] \\
&= \mathbb{E}[\mathbf{1}[X \leq x] \mathbb{P}[Y \leq s - \star]_{\star = X}] \\
&= \mathbb{E}\left[\mathbf{1}[X \leq x] \mathbf{1}[X \leq s] \left(1 - e^{-\lambda(s-X)}\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}[X \leq \min(x,s)] \left(1 - e^{-\lambda(s-X)}\right)\right] \\
&= \int_0^{\min(x,s)} \lambda e^{-\lambda t} \left(1 - e^{-\lambda(s-t)}\right) dt \\
&= \int_0^{\min(x,s)} \lambda e^{-\lambda t} dt - \int_0^{\min(x,s)} \lambda e^{-\lambda t} e^{-\lambda(s-t)} dt \\
&= 1 - e^{-\lambda \min(x,s)} - \int_0^{\min(x,s)} \lambda e^{-\lambda s} dt. \qquad (1.1)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{P}[X \leq x, X+Y \leq s] &= 1 - e^{-\lambda \min(x,s)} - \lambda \min(x,s) e^{-\lambda s} \\
&= \begin{cases} 1 - e^{-\lambda x} - \lambda x e^{-\lambda s} & \text{if } 0 \leq x \leq s \\ \\ 1 - e^{-\lambda s} - \lambda s e^{-\lambda s} & \text{if } s \leq x. \end{cases} \qquad (1.2)
\end{aligned}$$

1.b. The joint probability distribution function of $(X, X+Y)$ is differentiable everywhere except possibly on the line $s = x$ (which is a one-dimensional manifold in $\mathbb{R}^2$ and therefore has Lebesgue measure zero!) Therefore, this joint distribution is of continuous type with density obtained by taking the mixed deribvative with respect to x and s: We get

$$f_{X,X+Y}(x,s) = \frac{\partial^2}{\partial x \partial s} \mathbb{P}[X+Y \leq s, X \leq x]$$

when $s \neq x$ in $\mathbb{R}^2$. Thus,

$$f_{X,X+Y}(x,s) = \begin{cases} \lambda^2 e^{-\lambda s} & \text{if } 0 \leq x < s \\ \\ 0 & \text{if } s < x. \end{cases} \qquad (1.3)$$

1.c. It follows from the calculations above[1] that

$$\mathbb{P}\left[X+Y \le s\right] = 1 - e^{-\lambda s} - \lambda s e^{-\lambda s}, \quad s \ge 0$$

whence

$$f_{X+Y}(s) = \lambda^2 s e^{-\lambda s}, \quad s \ge 0.$$

Therefore,

$$\begin{aligned} f_{X|X+Y}(x|s) &= \frac{f_{X+Y,X}(x,s)}{f_{X+Y}(s)} \\ &= \begin{cases} \frac{\lambda^2 e^{-\lambda s}}{\lambda^2 s e^{-\lambda s}} & \text{if } 0 \le x < s \\ \\ 0 & \text{if } s < x. \end{cases} \end{aligned} \tag{1.4}$$

In short,

$$[X|X+Y=s] =_{st} \mathcal{U}(0,s).$$

1.d. In view of Part **1.c** we find

$$\mathbb{E}\left[X|X+Y=s\right] = \int_0^s x \cdot \frac{1}{s} dx = \frac{s}{2}, \quad s > 0.$$

This is not too surprising in view of the following reasoning: By symmetry we have

$$\mathbb{E}\left[X|X+Y=s\right] = \mathbb{E}\left[Y|X+Y=s\right]$$

so that

$$\begin{aligned} \mathbb{E}\left[X+Y|X+Y=s\right] &= \mathbb{E}\left[X|X+Y=s\right] + \mathbb{E}\left[Y|X+Y=s\right] \\ &= 2\mathbb{E}\left[X|X+Y=s\right] \end{aligned} \tag{1.5}$$

by linearity. Yet,

$$\mathbb{E}\left[X+Y|X+Y=s\right] = s$$

and the conclusion

$$\mathbb{E}\left[X|X+Y=s\right] = \mathbb{E}\left[Y|X+Y=s\right] = \frac{s}{2}$$

follows!

2. ____

2.a. Fix distinct $k, \ell = 1, \ldots, n$ so that $k+2$ and $\ell+2$ are also distinct. Note that

$$\begin{aligned} \text{Cov}[Z_k, Z_\ell] &= \text{Cov}[X_k - X_{k+2}, X_\ell - X_{\ell+2}] \\ &= \text{Cov}[X_k, X_\ell] - \text{Cov}[X_k, X_{\ell+2}] - \text{Cov}[X_{k+2}, X_\ell] + \text{Cov}[X_{k+2}, X_{\ell+2}] \\ &= -\text{Cov}[X_k, X_{\ell+2}] - \text{Cov}[X_{k+2}, X_\ell] \\ &= -\left(\delta(k, \ell+2) + \delta(k+2, \ell)\right)\sigma^2. \end{aligned} \tag{1.6}$$

[1] Just take $x \uparrow \infty$.

It is now plain that

$$\text{Cov}[Z_k, Z_\ell] = \begin{cases} 0 & \text{if } |k-\ell| \neq 2 \\ & \\ -\sigma^2 & \text{if } |k-\ell| = 2 \end{cases}$$

and only when $|k-\ell| = 2$ are the rvs Z_k and Z_ℓ *not* correlated!

2.b. Write

$$S_p = X_1 + \ldots + X_p, \quad p = 1, 2, \ldots, n+2$$

and

$$R_n = Z_1 + \ldots + Z_n.$$

Note that

$$R_n = X_1 + X_2 - X_{n+1} - X_{n+2}.$$

Therefore, if $n = 1$, then $R_1 = X_1 - X_3$ so that $\text{Var}[R_1] = 2\sigma^2$. On the other hand, if $n = 2, 3, \ldots$, then the four rvs X_1, X_2, X_{n+1} and X_{n+2} being uncorrelated, we get

$$\text{Var}[R_n] = \text{Var}[X_1 + X_2 - X_{n+1} - X_{n+2}] = 4\sigma^2$$

under the assumption that the rvs $\{X_1, \ldots, X_{n+2}\}$ are uncorrelated.

2.c. This time we find

$$\begin{aligned} \text{Cov}[S_n, R_n] &= \text{Cov}[S_n, X_1 + X_2 - X_{n+1} - X_{n+2}] \\ &= \text{Cov}[X_1 + \ldots + X_n, X_1 + X_2 - X_{n+1} - X_{n+2}] \\ &= \text{Cov}[X_1 + \ldots + X_n, X_1 + X_2] \\ &= \begin{cases} \sigma^2 & \text{if } n = 1 \\ & \\ 2\sigma^2 & \text{if } n = 2, 3, \ldots \end{cases} \end{aligned} \tag{1.7}$$

since the rvs $\{X_1, \ldots, X_{n+2}\}$ are uncorrelated.

3. ---

Throughout $u, w \geq 0$ are held fixed.

3.a. Note that

$$\begin{aligned} \mathbb{P}[X < Y] &= \mathbb{E}[\mathbf{1}[X < Y]] \\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}[X < Y] | X]] \\ &= \mathbb{E}[(\mathbb{E}[\mathbf{1}[\star < Y] | X = \star])_{\star = X}] \\ &= \mathbb{E}[(\mathbb{E}[\mathbf{1}[\star < Y]])_{\star = X}] \\ &= \mathbb{E}[(\mathbb{P}[\star < Y])_{\star = X}] \\ &= \mathbb{E}\left[e^{-\mu X}\right] \\ &= \int_0^\infty e^{-\mu x} \lambda e^{-\lambda x} dx \end{aligned} \tag{1.8}$$

so that

$$\mathbb{P}[X < Y] = \frac{\lambda}{\lambda + \mu}.$$

By symmetry, we also have

$$\mathbb{P}[Y < X] = \frac{\mu}{\lambda + \mu}.$$

3.b. Note that

$$\mathbb{P}[X = Y] = \int_0^\infty \mathbb{P}[X = t] \mu e^{-\mu t} dt = 0$$

under the assumed independence. Next, we start with the observation that

$$\begin{aligned}
&\mathbb{P}[U \le u, W \le w] \\
&= \mathbb{P}[X < Y, U \le u, W \le w] + \mathbb{P}[Y < X, U \le u, W \le w] \\
&= \mathbb{P}[X < Y, X \le u, Y - X \le w] + \mathbb{P}[Y < X, Y \le u, X - Y \le w]
\end{aligned} \tag{1.9}$$

Therefore, considering the fist term, we can use the independence of the rvs X and Y to find

$$\begin{aligned}
&\mathbb{P}[X < Y, X \le u, Y - X \le w] \\
&= \mathbb{E}[\mathbf{1}[X < Y, X \le u, Y - X \le w]] \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}[X < Y, X \le u, Y - X \le w] | X]] \\
&= \mathbb{E}[\mathbf{1}[X \le u] \mathbb{E}[\mathbf{1}[X < Y, Y - X \le w] | X]] \\
&= \mathbb{E}[\mathbf{1}[X \le u] (\mathbb{E}[\mathbf{1}[X < Y, Y - X \le w] | X = \star])_{\star = X}] \\
&= \mathbb{E}[\mathbf{1}[X \le u] (\mathbb{E}[\mathbf{1}[\star < Y, Y - \star \le w] | X = \star])_{\star = X}] \\
&= \mathbb{E}[\mathbf{1}[X \le u] (\mathbb{E}[\mathbf{1}[\star < Y, Y - \star \le w]])_{\star = X}] \\
&= \mathbb{E}[\mathbf{1}[X \le u] (\mathbb{E}[\mathbf{1}[\star < Y, Y \le \star + w]])_{\star = X}] \\
&= \mathbb{E}[\mathbf{1}[X \le u] (\mathbb{P}[\star < Y \le w + \star])_{\star = X}] \\
&= \mathbb{E}\left[\mathbf{1}[X \le u] \left(e^{-\mu X} - e^{-\mu(w+X)}\right)\right] \\
&= \left(1 - e^{-\mu w}\right) \mathbb{E}\left[\mathbf{1}[X \le u] e^{-\mu X}\right] \\
&= \left(1 - e^{-\mu w}\right) \int_0^u \lambda e^{-\lambda x} e^{-\mu x} dx
\end{aligned} \tag{1.10}$$

so that

$$\mathbb{P}[X < Y, X \le u, Y - X \le w] = \frac{\lambda}{\lambda + \mu} \left(1 - e^{-(\lambda+\mu)u}\right) \left(1 - e^{-\mu w}\right).$$

By symmetry, we also have

$$\mathbb{P}[Y < X, Y \le u, X - Y \le w] = \frac{\mu}{\lambda + \mu} \left(1 - e^{-(\lambda+\mu)u}\right) \left(1 - e^{-\lambda w}\right).$$

Combining the last two expressions, we conclude that

$$\mathbb{P}[U \le u, W \le w] = \left(1 - e^{-(\lambda+\mu)u}\right) \left(\frac{\lambda}{\lambda + \mu} \left(1 - e^{-\mu w}\right) + \frac{\mu}{\lambda + \mu} \left(1 - e^{-\lambda w}\right)\right)$$

3.c. Let u and w go to infinity in this last expression. This yields[2]

$$\mathbb{P}[U \le u] = 1 - e^{-(\lambda+\mu)u}, \quad u \ge 0$$

and

$$\mathbb{P}[W \le w] = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-\mu w}\right) + \frac{\mu}{\lambda+\mu}\left(1 - e^{-\lambda w}\right), \quad w \ge 0.$$

Therefore,

$$\mathbb{P}[U \le u, W \le w] = \mathbb{P}[U \le u]\,\mathbb{P}[W \le w], \quad u, w \ge 0$$

and the independence of the rvs U and W follows!

4. ____

4.a. One possible choice is to take $\Omega = \{0,1\}^n \times \{0,1\}^n$ with the following interpretation: With

$$\omega = (a_1, \ldots, a_n; b_1, \ldots, b_n)$$

a generic element of Ω where a_i, b_i in $\{0,1\}$ for each $i = 1, \ldots, n$, $a_i = 1$ (resp. $a_i = 0$) means that Alice has (resp. has not) selected item i, and $b_i = 1$ (resp. $b_i = 0$) means that Bob has (resp. has not) selected item i.

The set Ω being finite, we shall take $\mathcal{F}$ to be the power set of Ω, and define $\mathbb{P}$ by

$$\mathbb{P}[(a_1, \ldots, a_n; b_1, \ldots, b_n)] = \prod_{i=1}^{n} p^{a_i}(1-p)^{1-a_i} \cdot \prod_{i=1}^{n} p^{b_i}(1-p)^{1-b_i}.$$

4.b. For each $i = 1, \ldots, n$, it is convenient to define the rvs $A_i, B_i : \Omega \to \{0,1\}$ by

$$A_i(\omega) = a_i \quad \text{and} \quad B_i(\omega) = b_i$$

where $\omega = (a_1, \ldots, a_n; b_1, \ldots, b_n)$ as before. Under the assumptions of the problem, the rvs $\{A_i, B_i,\ i = 1, 2, \ldots, n\}$ are i.i.d. $\{0,1\}$-valued rvs with

$$\mathbb{P}[A_i = 1] = \mathbb{P}[B_i = 1] = p, \quad i = 1, \ldots, n.$$

Note that $A = \{i = 1, \ldots, n : \ A_i = 1\}$ and $B = \{i = 1, \ldots, n : \ B_i = 1\}$, so that

$$[A \cap B = \emptyset] = \cap_{i=1}^{n} [A_i B_i = 0].$$

By independence,

$$\mathbb{P}[A \cap B = \emptyset] = \prod_{i=1}^{n} \mathbb{P}[A_i B_i = 0]$$

with

$$\begin{aligned} \mathbb{P}[A_i B_i = 0] &= 1 - \mathbb{P}[A_i B_i = 1] \\ &= 1 - \mathbb{P}[A_i = 1]\,\mathbb{P}[B_i = 1] = 1 - p^2. \end{aligned} \tag{1.11}$$

[2] And this first conclusion should not surprise you since it is already known to you [Remember problem 6 in HW # 5] that U is exponentially distributed with parameter $\lambda + \mu$.

Collecting we conclude that

$$\mathbb{P}\left[A \cap B = \emptyset\right] = \left(1 - p^2\right)^n .$$

4.c. In the usual manner, we have

$$\mathbb{P}\left[A \cap B = \emptyset \;\middle|\; \begin{array}{c} |A| \neq 0 \\ |B| \neq 0 \end{array}\right] = \frac{\mathbb{P}\left[A \cap B = \emptyset, |A| \neq 0, |B| \neq 0\right]}{\mathbb{P}\left[|A| \neq 0, |B| \neq 0\right]}$$

It is plain that

$$\begin{aligned} \mathbb{P}\left[|A| \neq 0, |B| \neq 0\right] &= \mathbb{P}\left[|A| \neq 0\right] \mathbb{P}\left[|B| \neq 0\right] \\ &= \left(1 - \mathbb{P}\left[|A| = 0\right]\right)\left(1 - \mathbb{P}\left[|B| = 0\right]\right) \\ &= \left(1 - (1-p)^n\right)^2 \end{aligned} \tag{1.12}$$

On the other hand,

$$\begin{aligned} &\mathbb{P}\left[A \cap B = \emptyset, |A| \neq 0, |B| \neq 0\right] \\ &\quad = \mathbb{P}\left[A \cap B = \emptyset\right] - \mathbb{P}\left[A \cap B = \emptyset, |A| = 0, |B| \neq 0\right] \\ &\quad\quad - \mathbb{P}\left[A \cap B = \emptyset, |A| \neq 0, |B| = 0\right] - \mathbb{P}\left[A \cap B = \emptyset, |A| = |B| = 0\right] \\ &\quad = \mathbb{P}\left[A \cap B = \emptyset\right] - \mathbb{P}\left[|A| = 0, |B| \neq 0\right] \\ &\quad\quad - \mathbb{P}\left[|A| \neq 0, |B| = 0\right] - \mathbb{P}\left[|A| = |B| = 0\right] \end{aligned}$$

since $A \cap B = \emptyset$ as soon as either set A or B (or both) is empty.

Note that

$$\mathbb{P}\left[|A| = 0, |B| \neq 0\right] = \mathbb{P}\left[|A| = 0\right] \mathbb{P}\left[|B| \neq 0\right] = (1-p)^n \left(1 - (1-p)^n\right),$$

$$\mathbb{P}\left[|A| \neq 0, |B| = 0\right] = \mathbb{P}\left[|A| \neq 0\right] \mathbb{P}\left[|B| = 0\right] = (1-p)^n \left(1 - (1-p)^n\right)$$

and

$$\mathbb{P}\left[|A| = |B| = 0\right] = \mathbb{P}\left[|A| = 0\right] \mathbb{P}\left[|B| = 0\right] = (1-p)^n (1-p)^n = (1-p)^{2n}.$$

Thus,

$$\begin{aligned} &\mathbb{P}\left[A \cap B = \emptyset, |A| \neq 0, |B| \neq 0\right] \\ &\quad = \left(1 - p^2\right)^n - 2(1-p)^n \left(1 - (1-p)^n\right) - (1-p)^{2n} \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}\left[A \cap B = \emptyset \middle| \begin{array}{c} |A| \neq 0 \\ |B| \neq 0 \end{array}\right] \\ &\quad = \frac{\left(1 - p^2\right)^n - 2(1-p)^n \left(1 - (1-p)^n\right) - (1-p)^{2n}}{\left(1 - (1-p)^n\right)^2}. \end{aligned}$$
