ENEE 620
FALL 2011
RANDOM PROCESSES
IN COMMUNICATION AND CONTROL

ANSWER KEY TO TEST # 2:

1. ______

1.a. Since $\boldsymbol{R}$ is symmetric we need only show that it is positive semi-definite. We write any element $\boldsymbol{v}$ in $\mathbb{R}^{n+1}$ as

$$\boldsymbol{v} = \begin{bmatrix} x \\ \boldsymbol{y} \end{bmatrix}$$

with x in $\mathbb{R}$ and $\boldsymbol{y} = (y_1, \ldots, y_n)'$ in $\mathbb{R}^n$. With this notation we get

$$\boldsymbol{Rv} = \begin{bmatrix} \sigma^2 x + \sum_{k=1}^n \rho_k y_k \\ \rho_1 x + \sigma_1^2 y_1 \\ \rho_2 x + \sigma_2^2 y_2 \\ \vdots \\ \rho_k x + \sigma_k^2 y_k \\ \vdots \\ \rho_n x + \sigma_n^2 y_n \end{bmatrix}$$

so that

$$\begin{aligned} \boldsymbol{v}'\boldsymbol{Rv} &= \left(\sigma^2 x + \sum_{k=1}^n \rho_k y_k \right) x + \sum_{k=1}^n y_k \left(\rho_k x + \sigma_k^2 y_k \right) \\ &= \sigma^2 x^2 + 2 \sum_{k=1}^n \rho_k x y_k + \sum_{k=1}^n \sigma_k^2 y_k^2 \\ &= \sigma^2 x^2 + \sum_{k=1}^n \left(\sigma_k^2 y_k^2 + 2\rho_k x y_k \right) \\ &= \sigma^2 x^2 + \sum_{k=1}^n \left(\sigma_k^2 y_k^2 + 2 \left(\frac{\rho_k}{\sigma_k} x \right) (\sigma_k y_k) \right) \\ &= \sigma^2 x^2 + \sum_{k=1}^n \left(\frac{\rho_k}{\sigma_k} x + \sigma_k y_k \right)^2 - \sum_{k=1}^n \left(\frac{\rho_k}{\sigma_k} x \right)^2 \\ &= \left(\sigma^2 - \sum_{k=1}^n \left(\frac{\rho_k}{\sigma_k} \right)^2 \right) x^2 + \sum_{k=1}^n \left(\frac{\rho_k}{\sigma_k} x + \sigma_k y_k \right)^2 \qquad (1.1) \end{aligned}$$

It is now plain that $\boldsymbol{R}$ is positive semi-definite, i.e., $\boldsymbol{v}'\boldsymbol{Rv} \geq 0$ for *all* $\boldsymbol{v}$ in $\mathbb{R}^{n+1}$, if and only if

$$\sigma^2 - \sum_{k=1}^n \left(\frac{\rho_k}{\sigma_k} \right)^2 \geq 0.$$

This condition is now enforced on the parameters entering $\boldsymbol{R}$.

Many of you used a criterion for positive semi-definiteness that involves the leading principal minors. Unfortunately, this condition is necessary and sufficient for positive definiteness, but only necessary for semi-positive definiteness.

1.b. The $(n+1)$-dimensional random vector $(X, Y_1, \ldots, Y_n)'$ is assumed to be normally distributed $\mathrm{N}((0,0,\ldots,0)', \boldsymbol{R})$. The existence of a probability density function is equivalent to $\boldsymbol{R}$ being invertible (i.e., positive definite), and this occurs if and only if

$$\sigma^2 - \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k}\right)^2 > 0.$$

Indeed, $\boldsymbol{v}'\boldsymbol{R}\boldsymbol{v} = 0$ occurs if and only if

$$\left(\sigma^2 - \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k}\right)^2\right) x^2 = 0$$

and

$$\left(\frac{\rho_k}{\sigma_k} x + \sigma_k y_k\right)^2 = 0, \quad k = 1, \ldots, n$$

If the condition $\boldsymbol{v}'\boldsymbol{R}\boldsymbol{v} = 0$ must imply $\mathbf{0}_{n+1}$, then we necessarily have

$$\left(\sigma^2 - \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k}\right)^2\right) \neq 0$$

and the announced condition follows!

1.c. By Part **1.b**, the probability distribution function of the $(n+1)$-dimensional random vector $(X, Y_1, \ldots, Y_n)'$ will not admit a probability density function if and only if

$$\sigma^2 - \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k}\right)^2 = 0,$$

in which case

$$\boldsymbol{v}'\boldsymbol{R}\boldsymbol{v} = \sum_{k=1}^{n} \left(\frac{\rho_k}{\sigma_k} x + \sigma_k y_k\right)^2, \quad \boldsymbol{v} \in \mathbb{R}^{n+1}.$$

Therefore, $\boldsymbol{v}'\boldsymbol{R}\boldsymbol{v} = 0$ if and only if

$$\frac{\rho_k}{\sigma_k} x + \sigma_k y_k = 0, \quad k = 1, 2, \ldots, n$$

i.e.,

$$y_k = -\left(\frac{\rho_k}{\sigma_k^2}\right) x, \quad k = 1, 2, \ldots, n$$

This suggests introducing the linear subspace K of $\mathbb{R}^{n+1}$ given by

$$K = \{\boldsymbol{v} \in \mathbb{R}^{n+1} : \ \boldsymbol{v} = x\boldsymbol{a}, \ x \in \mathbb{R}\} = \mathbb{R}\boldsymbol{a}$$

where

$$\boldsymbol{a} = \left(1, -\left(\frac{\rho_1}{\sigma_1^2}\right), \ldots, -\left(\frac{\rho_n}{\sigma_n^2}\right)\right)' (\neq \boldsymbol{0}_{n+1}).$$

Note that $\dim(K) = 1$.

It now follows in the usual manner that

$$\begin{aligned}\mathrm{Var}\left[xX + \boldsymbol{y}'\boldsymbol{Y}\right] &= \boldsymbol{v}'\boldsymbol{R}\boldsymbol{v} \\ &= \sum_{k=1}^{n}\left(\frac{\rho_k}{\sigma_k}x + \sigma_k y_k\right)^2 \\ &= 0, \quad \boldsymbol{v} \in K \end{aligned} \tag{1.2}$$

in which case

$$xX + \boldsymbol{y}'\boldsymbol{Y} = 0 \quad a.s.$$

whenever $\boldsymbol{v} = (x, \boldsymbol{y}')'$ is an element of K. Thus,

$$\mathbb{P}\left[(X, Y_1, \ldots, Y_n)' \in H\right] = 1$$

where

$$\begin{aligned} H &= K^{\perp} \\ &= \left\{\boldsymbol{v} \in \mathbb{R}^{n+1} : \boldsymbol{v}'\boldsymbol{a} = 0\right\} \\ &= \left\{(x, \boldsymbol{y}')' \in \mathbb{R}^{n+1} : \ x = \sum_{k=1}^{n}\frac{\rho_k}{\sigma_k^2}y_k\right\} \end{aligned} \tag{1.3}$$

and $\dim(K) = n$.

2.

2.a. For each t in $\mathbb{R}$, note that

$$\mathbb{E}\left[e^{itX}\right] = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}e^{ikt} = e^{-\lambda(1-e^{it})}$$

by standard calculations, with a similar expression for $\mathbb{E}\left[e^{itY}\right]$. By the independence of the rvs X and Y, hence of the rvs e^{itX} and e^{-itY}, we get

$$\begin{aligned}\mathbb{E}\left[e^{it(X-Y)}\right] &= \mathbb{E}\left[e^{itX}\right]\cdot\mathbb{E}\left[e^{-itY}\right] \\ &= e^{-\lambda(1-e^{it})}\cdot e^{-\lambda(1-e^{-it})} \\ &= e^{-2\lambda(1-\cos t)}\end{aligned} \tag{1.4}$$

as we recall the identity

$$\cos t = \frac{e^{it} + e^{-it}}{2}.$$

2.b. Fix $n = 1, 2, \ldots$ and t in $\mathbb{R}$. For each $k = 1, 2, \ldots, n$, we find

$$\begin{aligned}
\mathbb{E}\left[e^{it\frac{X_{n,k}}{\sqrt{n}}}\right] &= \frac{1}{2n}\left(e^{it} + e^{-it}\right) + 1 - \frac{1}{n} \\
&= \frac{1}{n}\cos t + 1 - \frac{1}{n} \\
&= \left(1 - \frac{1}{n}(1 - \cos t)\right) \qquad (1.5)
\end{aligned}$$

Using the independence of the rvs $X_{n,1}, \ldots, X_{n,n}$ we get

$$\begin{aligned}
\mathbb{E}\left[e^{it\frac{S_n}{\sqrt{n}}}\right] &= \prod_{k=1}^{n} \mathbb{E}\left[e^{it\frac{X_{n,k}}{\sqrt{n}}}\right] \\
&= \left(1 - \frac{1}{n}(1 - \cos t)\right)^n. \qquad (1.6)
\end{aligned}$$

2.c. It is now plain that

$$\lim_{n\to\infty} \mathbb{E}\left[e^{it\frac{S_n}{\sqrt{n}}}\right] = \lim_{n\to\infty} \left(1 - \frac{1}{n}(1 - \cos t)\right)^n = e^{-(1-\cos t)}$$

for each t in $\mathbb{R}$, thus $\frac{S_n}{\sqrt{n}} \Longrightarrow_n L$ where L is distributed like the difference of two independent Poisson rvs with parameter $\lambda = \frac{1}{2}$ – This is an immediate consequence of Part **2.a**.

3. ____

3.a. There are several different ways to show that

$$\lim_{n\to\infty} \frac{X_n}{\sqrt[p]{n}} = 0 \quad a.s. \qquad (1.7)$$

for each $p \geq 1$.

First approach – The rvs $\{X, X_n,\ n = 1, 2, \ldots\}$ are i.i.d. rvs, each exponentially distributed with unit parameter. Thus,

$$\mathbb{E}\left[X^k\right] < \infty, \quad k = 1, 2, \ldots \qquad (1.8)$$

so that $\mathbb{E}[X^p] < \infty$ for each $p \geq 1$ – Just apply (1.8) with $k(p) = \lceil p \rceil$ and use the fact that $\mathbb{E}[X^p]$ is necessarily finite since $p < \lceil p \rceil$.

The rvs $\{(X_n)^p,\ n = 1, 2, \ldots\}$ are still independent and identically distributed, and by the Strong Law of Large Numbers applied to this sequence of rvs we conclude that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (X_k)^p = \mathbb{E}[X^p] \quad a.s.$$

whence

$$\lim_{n\to\infty} \frac{(X_n)^p}{n} = 0 \quad a.s.$$

by standard arguments. This establishes (1.7).

Second approach – For every $\varepsilon > 0$, note that

$$\mathbb{P}\left[\frac{X_n}{\sqrt[p]{n}} > \varepsilon\right] = \mathbb{P}\left[X_n > \varepsilon \sqrt[p]{n}\right] = e^{-\varepsilon \sqrt[p]{n}}$$

for each $n = 1, 2, \ldots$. Thus,

$$\sum_{n=1}^{\infty} \mathbb{P}\left[\frac{X_n}{\sqrt[p]{n}} > \varepsilon\right] = \sum_{n=1}^{\infty} e^{-\varepsilon \sqrt[p]{n}} < \infty$$

and the a.s. convergence (1.7) follows.

3.b. For each $n = 1, 2, \ldots$, elementary calculations yield

$$\begin{aligned} R_n &= \sqrt{T_{n+1}} - \sqrt{T_n} \\ &= \frac{\left(\sqrt{T_{n+1}} - \sqrt{T_n}\right) \cdot \left(\sqrt{T_{n+1}} + \sqrt{T_n}\right)}{\sqrt{T_{n+1}} + \sqrt{T_n}} \\ &= \frac{T_{n+1} - T_n}{\sqrt{T_{n+1}} + \sqrt{T_n}} \\ &= \frac{X_{n+1}}{\sqrt{T_{n+1}} + \sqrt{T_n}} \\ &= \frac{\frac{X_{n+1}}{\sqrt{n+1}}}{\sqrt{\frac{T_{n+1}}{n+1}} + \sqrt{\frac{T_n}{n+1}}}. \end{aligned} \tag{1.9}$$

By the Strong of Large Numbers, we have

$$\lim_{n\to\infty} \frac{T_{n+1}}{n+1} = 1 \quad a.s. \quad \text{and} \quad \lim_{n\to\infty} \frac{T_n}{n+1} = 1 \quad a.s.$$

Using these facts and the convergence (1.7) (for $p = 2$) we find

$$\lim_{n\to\infty} R_n = 0 \quad a..s.$$

4. ____

4.a. Here the Orthogonality Principle reads

$$\mathbb{E}\left[(X - Z^\star)\, Z\right] = 0, \quad Z \in V. \tag{1.10}$$

However, any element Z of V is of the form

$$Z = \sum_{k=1}^{n} a_k Y_k + b$$

with arbitrary $a_1, \ldots, a_n$ and b in $\mathbb{R}$. Obviously, the rvs $Z = Y_1, \ldots, Z = Y_n$ and $Z = 1$ are in V. Using them in (1.10) we get

$$\mathbb{E}\left[(X - Z^\star)\, 1\right] = 0 \quad \text{and} \quad \mathbb{E}\left[(X - Z^\star)\, Y_k\right] = 0, \quad k = 1, \ldots, n.$$

Thus, $Z^\star$ satisfies

$$\mathbb{E}[X] = \mathbb{E}[Z^\star] \quad \text{and} \quad \mathbb{E}[(X - \mathbb{E}[X] + \mathbb{E}[Z^\star] - Z^\star) Y_k] = 0, \quad k = 1, \ldots, n$$

and this is equivalent to

$$\mathbb{E}[X] = \mathbb{E}[Z^\star] \quad \text{and} \quad \text{Cov}[X - Z^\star, Y_k] = 0, \quad k = 1, \ldots, n. \tag{1.11}$$

By linearity it is elementary to see that (1.10) and (1.11) are indeed equivalent.

4.b. Recall that $Z^\star$ is of the form

$$Z^\star = \sum_{\ell=1}^{n} a_\ell^\star Y_\ell + b^\star$$

with $a_1^\star, \ldots, a_n^\star$ and $b^\star$ in $\mathbb{R}$ such that (1.11) holds. In particular,

$$\mathbb{E}[X] = \sum_{\ell=1}^{n} a_\ell^\star \mathbb{E}[Y_\ell] + b^\star$$

and for each $k = 1, 2, \ldots, n$, we find

$$\begin{aligned}
\text{Cov}[X - Z^\star, Y_k] &= \text{Cov}[X - \left(\sum_{\ell=1}^{n} a_\ell^\star Y_\ell + b^\star \right), Y_k] \\
&= \text{Cov}[X, Y_k] - \sum_{\ell=1}^{n} a_\ell^\star \text{Cov}[Y_\ell, Y_k] \\
&= \text{Cov}[X, Y_k] - a_k^\star \text{Var}[Y_k] \\
&= \rho_k - a_k^\star \sigma_k^2
\end{aligned} \tag{1.12}$$

whence

$$a_k^\star = \frac{\rho_k}{\sigma_k^2}.$$

It is now plain that

$$\begin{aligned}
Z^\star &= \sum_{\ell=1}^{n} a_\ell^\star Y_\ell + b^\star \\
&= \sum_{\ell=1}^{n} a_\ell^\star Y_\ell + \left(\mathbb{E}[X] - \sum_{\ell=1}^{n} a_\ell^\star \mathbb{E}[Y_\ell] \right) \\
&= \mathbb{E}[X] + \sum_{\ell=1}^{n} a_\ell^\star (Y_\ell - \mathbb{E}[Y_\ell]) \\
&= \mathbb{E}[X] + \sum_{\ell=1}^{n} \frac{\rho_\ell}{\sigma_\ell^2} (Y_\ell - \mathbb{E}[Y_\ell]).
\end{aligned} \tag{1.13}$$

4.c. Next, we note that

$$X - Z^\star = X - \mathbb{E}\left[X\right] - \sum_{\ell=1}^{n} \frac{\rho_\ell}{\sigma_\ell^2}\left(Y_\ell - \mathbb{E}\left[Y_\ell\right]\right)$$

so that

$$\begin{aligned}
\mathrm{Var}[X - Z^\star] &= \mathbb{E}\left[(X - Z^\star)^2\right] \\
&= \mathbb{E}\left[\left(X - \mathbb{E}\left[X\right] - \sum_{\ell=1}^{n} \frac{\rho_\ell}{\sigma_\ell^2}\left(Y_\ell - \mathbb{E}\left[Y_\ell\right]\right)\right)^2\right] \\
&= \mathrm{var}[X] + \sum_{\ell=1}^{n}\left(\frac{\rho_\ell}{\sigma_\ell^2}\right)^2 \sigma_\ell^2 - 2\sum_{\ell=1}^{n}\frac{\rho_\ell}{\sigma_\ell^2}\mathrm{Cov}[X, Y_\ell] \\
&= \mathrm{var}[X] + \sum_{\ell=1}^{n}\left(\frac{\rho_\ell}{\sigma_\ell}\right)^2 - 2\sum_{\ell=1}^{n}\left(\frac{\rho_\ell}{\sigma_\ell}\right)^2 \\
&= \mathrm{var}[X] - \sum_{\ell=1}^{n}\left(\frac{\rho_\ell}{\sigma_\ell}\right)^2 . \qquad (1.14)
\end{aligned}$$
