ENEE 620
FALL 2011
RANDOM PROCESSES
IN COMMUNICATION AND CONTROL

ANSWER KEY TO FINAL EXAM

1.

Let $\mathcal{P}_K$ denote the collection of subsets of $\{1, \ldots, P\}$ of size exactly K. Obviously,

$$|\mathcal{P}_K| = \binom{P}{K}$$

1.a. Pick a non-empty subset S of $\{1, \ldots, P\}$. By the uniform assumption on K_i we have

$$\mathbb{P}[K_i = T] = \frac{1}{\binom{P}{K}}, \quad T \in \mathcal{P}_K$$

so that

$$\begin{aligned} q(S) &= \mathbb{P}[K_i \cap S = \emptyset] \\ &= \sum_{T \in \mathcal{P}_K:\ T \cap S = \emptyset} \mathbb{P}[K_i = T] \\ &= \sum_{T \in \mathcal{P}_K:\ T \cap S = \emptyset} \frac{1}{\binom{P}{K}} \\ &= \frac{|\{T \in \mathcal{P}_K :\ T \cap S = \emptyset\}|}{\binom{P}{K}} \end{aligned} \tag{1.1}$$

with

$$|\{T \in \mathcal{P}_K :\ T \cap S = \emptyset\}| = \begin{cases} \binom{P-|S|}{K} & \text{if } |S| + K \leq P \\ 0 & \text{if } P < |S| + K. \end{cases}$$

Collecting we find,

$$\mathbb{P}[K_i \cap S = \emptyset] = \begin{cases} \frac{\binom{P-|S|}{K}}{\binom{P}{K}} & \text{if } |S| + K \leq P \\ 0 & \text{if } P < |S| + K. \end{cases}$$

1.b. Here we have $2K \leq P$. Using the law of total probabilities we get

$$\begin{aligned} \mathbb{P}[K_i \cap K_j = \emptyset] &= \mathbb{E}[\mathbf{1}[K_i \cap K_j = \emptyset]] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{E}[\mathbf{1}[K_i = S]\,\mathbf{1}[K_i \cap K_j = \emptyset]] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{E}[\mathbf{1}[K_i = S]\,\mathbf{1}[S \cap K_j = \emptyset]] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{E}[\mathbf{1}[K_i = S]]\,\mathbb{E}[\mathbf{1}[S \cap K_j = \emptyset]] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{P}[K_i = S]\,\mathbb{P}[S \cap K_j = \emptyset] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{P}[K_i = S]\,q(|S|) \\ &= q(K) \sum_{S \in \mathcal{P}_K} \mathbb{P}[K_i = S] \qquad (1.2) \end{aligned}$$

upon invoking the independence of the random sets K_i and K_j and using Part **1.a**. In conclusion,

$$\mathbb{P}[\chi_{ij} = 1] = \mathbb{P}[K_i \cap K_j = \emptyset] = q(K). \qquad (1.3)$$

1.c. Pick $b_2, \ldots, b_n$ elements in $\{0, 1\}$, and write

$$\chi_{1j} = 1 \quad \text{if and only if} \quad K_1 \cap K_j =_{b_j} \emptyset$$

With this notation we find

$$\begin{aligned} \mathbb{P}[\chi_{1j} = b_j,\ j = 2, \ldots, n] &= \mathbb{P}\left[K_1 \cap K_j =_{b_j} \emptyset,\ j = 2, \ldots, n\right] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{E}\left[\mathbf{1}[K_1 = S]\,\mathbf{1}\left[K_1 \cap K_j =_{b_j} \emptyset,\ j = 2, \ldots, n\right]\right] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{E}\left[\mathbf{1}[K_1 = S]\,\mathbf{1}\left[S \cap K_j =_{b_j} \emptyset,\ j = 2, \ldots, n\right]\right] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{P}[K_1 = S]\,\mathbb{P}\left[S \cap K_j =_{b_j} \emptyset,\ j = 2, \ldots, n\right] \\ &= \sum_{S \in \mathcal{P}_K} \mathbb{P}[K_1 = S] \prod_{j=2}^{n} q(K)^{b_j} (1 - q(K))^{1-b_j} \\ &= q(K)^b (1 - q(K))^{n-1-b} \qquad (1.4) \end{aligned}$$

under the enforced i.i.d. assumptions, with

$$b = b_2 + \ldots + b_n.$$

1.d. It is clear from Part **1.c** the rvs $\chi_{12}, \chi_{13}, \ldots, \chi_{1n}$ are mutually independent.

2.

2.a. Yes as the needed sample path properties of $\{N^\star_t,\ t \geq 0\}$ are inherited from those of $\{N_t,\ t \geq 0\}$.

2.b. Fix $n = 1, 2, \ldots$ and $0 < t_1, \ldots < t_n$. For $x_1, \ldots, x_n$ arbitrary non-negative integers, we have

$$\begin{aligned}
&\mathbb{P}\left[N^\star_{t_1} = x_1, N^\star_{t_2} - N^\star_{t_1} = x_2, \ldots, N^\star_{t_n} - N^\star_{t_{n-1}} = x_n\right] \\
&= \mathbb{P}\left[N_{Yt_1} = x_1, N_{Yt_2} - N_{Yt_1} = x_2, \ldots, N_{Yt_n} - N_{Yt_{n-1}} = x_n\right] \\
&= \mathbb{E}\left[\mathbf{1}\left[N_{Yt_1} = x_1, N_{Yt_2} - N_{Yt_1} = x_2, \ldots, N_{Yt_n} - N_{Yt_{n-1}} = x_n\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left[N_{yt_1} = x_1, N_{yt_2} - N_{yt_1} = x_2, \ldots, N_{yt_n} - N_{yt_{n-1}} = x_n\right]\right]_{y=Y}\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[N_{yt_1} = x_1, N_{yt_2} - N_{yt_1} = x_2, \ldots, N_{yt_n} - N_{yt_{n-1}} = x_n\right]_{y=Y}\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[N_{yt_1} = x_1\right]_{y=Y} \prod_{j=2}^{n} \mathbb{P}\left[N_{yt_j} - N_{yt_{j-1}} = x_j\right]_{y=Y}\right] \\
&= \mathbb{E}\left[\frac{(\lambda Y t_1)^{x_1}}{x_1!} e^{-\lambda Y t_1} \prod_{j=2}^{n} \frac{(\lambda Y (t_j - t_{j-1}))^{x_j}}{x_j!} e^{-\lambda Y (t_j - t_{j-1})}\right] \qquad (1.5)
\end{aligned}$$

as we make use of the fact that the Poisson process $\{N_t,\ t \geq 0\}$ has mutually independent increments. However, the process $\{N^\star_t,\ t \geq 0\}$ does *not* have mutually independent increments. It only has mutually independent increments *conditionally* on the value of Y!

It should be clear that for each $t > 0$ we have

$$\mathbb{P}\left[N^\star_t = x\right] = \mathbb{E}\left[\frac{(\lambda Y t)^x}{x!} e^{-\lambda Y t}\right], \quad x = 0, 1, \ldots$$

and if Y is a non-degenerate rv, there is now way that this pmf is a Poisson pmf for some deterministic parameter $\lambda^\star$!

2.c. Fix $0 < s \leq t$. By the enforced independence, we have

$$\begin{aligned}
\mathbb{E}\left[N^\star_t\right] &= \mathbb{E}\left[N_{Yt}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[N_{yt}\right]_{y=Y}\right] \\
&= \mathbb{E}\left[\lambda(Yt)\right] = \lambda t \mathbb{E}\left[Y\right] \qquad (1.6)
\end{aligned}$$

In a similar way, we find

$$\begin{aligned}
\mathbb{E}\left[N^\star_s N^\star_t\right] &= \mathbb{E}\left[N_{Ys} N_{Yt}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[N_{ys} N_{yt}\right]_{y=Y}\right] \qquad (1.7)
\end{aligned}$$

with

$$\begin{aligned}
\mathbb{E}\left[N_{ys}N_{yt}\right] &= \mathbb{E}\left[N_{ys}\left(N_{yt}-N_{ys}\right)\right]+\mathbb{E}\left[N_{ys}^2\right] \\
&= \mathbb{E}\left[N_{ys}\right]\mathbb{E}\left[N_{yt}-N_{ys}\right]+\mathbb{E}\left[N_{ys}^2\right] \\
&= \lambda ys\cdot\lambda y(t-s)+\mathbb{E}\left[N_{ys}\right]+(\lambda ys)^2 \\
&= \lambda ys\cdot\lambda y(t-s)+\lambda ys+(\lambda ys)^2 \\
&= \lambda ys\cdot\lambda yt+\lambda ys
\end{aligned} \tag{1.8}$$

Therefore,

$$\mathbb{E}\left[N_t^\star\right]=\lambda^2 st\cdot\mathbb{E}\left[Y^2\right]+\lambda s\mathbb{E}\left[Y\right].$$

3.

Note that the rvs $\{\mathbf{1}\left[U_k\le B\right],\ k=1,2,\ldots\}$ are *not* i.i.d. but only *conditionally* i.i.d. *given* B!

3.a. Fix $n=1,2,\ldots$ and θ in $\mathbb{R}$. By independence we get

$$\begin{aligned}
\mathbb{E}\left[e^{i\theta S_n}\right] &= \mathbb{E}\left[e^{i\theta\sum_{k=1}^n\mathbf{1}[U_k\le B]}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{i\theta\sum_{k=1}^n\mathbf{1}[U_k\le B]}\middle|B\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{i\theta\sum_{k=1}^n\mathbf{1}[U_k\le b]}\middle|B=b\right]_{B=b}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{i\theta\sum_{k=1}^n\mathbf{1}[U_k\le b]}\right]_{B=b}\right]
\end{aligned} \tag{1.9}$$

where for each b in $[0,1]$, we have

$$\begin{aligned}
\mathbb{E}\left[e^{i\theta\sum_{k=1}^n\mathbf{1}[U_k\le b]}\right] &= \mathbb{E}\left[\prod_{k=1}^n e^{i\theta\mathbf{1}[U_k\le b]}\right] \\
&= \prod_{k=1}^n\mathbb{E}\left[e^{i\theta\mathbf{1}[U_k\le b]}\right] \\
&= \prod_{k=1}^n\left(1-b+be^{i\theta}\right) \\
&= \left(1-b+be^{i\theta}\right)^n
\end{aligned} \tag{1.10}$$

upon making use of the enforced independence assumptions.

As a result,

$$\begin{aligned}
\mathbb{E}\left[e^{i\theta\frac{S_n}{n}}\right] &= \mathbb{E}\left[\left(1-B+Be^{i\frac{\theta}{n}}\right)^n\right] \\
&= \mathbb{E}\left[\left(1-B\left(1-e^{i\frac{\theta}{n}}\right)\right)^n\right] \\
&= \mathbb{E}\left[\left(1-\frac{B}{n}\frac{1-e^{i\frac{\theta}{n}}}{\frac{1}{n}}\right)^n\right].
\end{aligned} \tag{1.11}$$

Since

$$\lim_{n\to\infty} B\left(\frac{1-e^{i\frac{\theta}{n}}}{\frac{1}{n}}\right) = -iB\theta,$$

it follows that ting that

$$\lim_{n\to\infty}\left(1-\frac{B}{n}\frac{1-e^{i\frac{\theta}{n}}}{\frac{1}{n}}\right)^n = e^{iB\theta}$$

so that

$$\lim_{n\to\infty}\mathbb{E}\left[e^{i\theta\frac{S_n}{n}}\right] = \mathbb{E}\left[e^{iB\theta}\right].$$

Upon identifying $\mathbb{E}\left[e^{iB\theta}\right]$ as the characteristic function of B, we conclude that $\frac{S_n}{n} \Longrightarrow_n B$ by the characterization of distributional convergence in terms of characteristic functions.

3.b. By an easy preconditioning argument, we get

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{S_n}{n}-B\right)^2\right] &= \mathbb{E}\left[\left(\frac{1}{n}\sum_{k=1}^n \mathbf{1}\left[U_k \le B\right]-B\right)^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{1}{n}\sum_{k=1}^n \mathbf{1}\left[U_k \le B\right]-B\right)^2 \Big| B\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{1}{n}\sum_{k=1}^n \mathbf{1}\left[U_k \le b\right]-b\right)^2 \Big| B=b\right]_{B=b}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\frac{1}{n}\sum_{k=1}^n \mathbf{1}\left[U_k \le b\right]-b\right)^2\right]_{B=b}\right] \\
&= \mathbb{E}\left[\left(\frac{b(1-b)}{n}\right)_{B=b}\right] \\
&= \frac{\mathbb{E}\left[B(1-B)\right]}{n} \qquad (1.12)
\end{aligned}$$

by standard calculations since for each b in $[0,1]$ it is plain that the rvs $\{\mathbf{1}\left[U_n \le b\right],\ n = 1,2,\ldots\}$ are i.i.d. Bernoulli rvs with first second moment b and variance $b(1-b)$.

Therefore,

$$\lim_{n\to\infty}\mathbb{E}\left[\left(\frac{S_n}{n}-B\right)^2\right] = 0,$$

whence $\frac{S_n}{n} \stackrel{L^2}{\rightarrow}_n B$ and the conclusion $\frac{S_n}{n} \stackrel{P}{\rightarrow}_n B$ follows.

4. ______________________________

4.a. Obviously, since $0 \leq D \leq 1$, we have $\mathbb{P}[D \leq t] = 0$ if $t < 0$ and $\mathbb{P}[D \leq t] = 1$ if $1 < t$. Thus, fix t in $[0,1]$. Using the independence of the rvs U and V we note that

$$\begin{aligned}
\mathbb{P}[D \leq t] &= \mathbb{E}[\mathbf{1}[|U-V| \leq t]] \\
&= \mathbb{E}[\mathbb{E}[\mathbf{1}[|U-V| \leq t]\,|V]] \\
&= \mathbb{E}[\mathbb{P}[|U-v| \leq t|V=v]_{v=V}] \\
&= \mathbb{E}[\mathbb{P}[|U-v| \leq t]_{v=V}]
\end{aligned}$$

where for each v in $[0,1]$,

$$\begin{aligned}
\mathbb{P}[|U-v| \leq t] &= \mathbb{P}[v-t \leq U \leq v+t] \\
&= \min(t+v,1) - \max(v-t,0) \\
&= \min(t+v,1-t+t) - \max(v-t,t-t) \\
&= (t+\min(v,1-t)) - (\max(v,t)-t) \\
&= 2t + \min(v,1-t) - \max(v,t). \qquad (1.13)
\end{aligned}$$

Therefore,

$$\mathbb{P}[D \leq t] = 2t + \mathbb{E}[\min(V,1-t)] - \mathbb{E}[\max(V,t)].$$

Next,

$$\begin{aligned}
\mathbb{E}[\min(V,1-t)] &= \int_0^{1-t} v\,dv + \int_{1-t}^1 (1-t)dv \\
&= \frac{(1-t)^2}{2} + t(1-t) \\
&= \frac{1-t^2}{2}. \qquad (1.14)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}[\max(V,t)] &= \int_0^t t\,dv + \int_t^1 v\,dv \\
&= t^2 + \frac{1-t^2}{2} \\
&= \frac{1+t^2}{2}. \qquad (1.15)
\end{aligned}$$

Collecting we conclude that

$$\begin{aligned}
\mathbb{P}[D \leq t] &= 2t + \mathbb{E}[\min(V,1-t)] - \mathbb{E}[\max(V,t)] \\
&= 2t + \frac{1-t^2}{2} - \frac{1+t^2}{2}. \\
&= 2t - t^2 \\
&= 1-(1-t)^2. \qquad (1.16)
\end{aligned}$$

4.b. Obviously we need only consider s in $[0,1]$, in which case we have

$$\begin{aligned} \mathbb{P}\left[S > s\right] &= \mathbb{P}\left[D > s, 1 - D > s\right] \\ &= \mathbb{P}\left[s < D < 1 - s\right]. \end{aligned} \tag{1.17}$$

First, we see that we need $s < 1 - s$. Hence,

$$\mathbb{P}\left[S > s\right] = 0, \quad \frac{1}{2} < s \leq 1$$

while on the range $0 \leq s \leq \frac{1}{2}$, we get

$$\begin{aligned} \mathbb{P}\left[S > s\right] &= \mathbb{P}\left[s < D < 1 - s\right] \\ &= \left(1 - (1 - (1 - s))^2\right) - \left(1 - (1 - s)^2\right) \\ &= (1 - s)^2 - s^2 \\ &= 1 - 2s. \end{aligned} \tag{1.18}$$

Collecting we conclude that

$$\mathbb{P}\left[S \leq s\right] = \min(1, 2s), \quad 0 \leq s \leq 1.$$

In short, S is uniformly distributed on the interval $[0, \frac{1}{2}]$.

4.c. Since

$$\mathbb{E}\left[\max(D, 1 - D)\right] + \mathbb{E}\left[\min(D, 1 - D)\right] = \mathbb{E}\left[D + (1 - D)\right] = 1,$$

we conclude that

$$\begin{aligned} \mathbb{E}\left[\max(D, 1 - D)\right] &= 1 - \mathbb{E}\left[S\right] \\ &= 1 - \int_0^{\frac{1}{2}} 2s ds \\ &= 1 - \frac{1}{4} = \frac{3}{4}. \end{aligned} \tag{1.19}$$

5. ____

First some easy observations: For each $n = 1, 2, \ldots$, write

$$\Xi_n = \xi_1 + \ldots + \xi_n.$$

and

$$\begin{aligned} S_n &= (\xi_1 + \ldots + \xi_n) + 2\left(\xi_2 + \ldots + \xi_{n+1}\right) \\ &= \xi_1 + 3\left(\xi_2 + \ldots + \xi_n\right) + 2\xi_{n+1}. \end{aligned} \tag{1.20}$$

Note that

$$S_n = \xi_1 + 3\left(\Xi_n - \xi_1\right) + 2\xi_{n+1}.$$

5.a. For each $n = 1, 2, \ldots$, we have

$$S_n \sim \mathrm{N}(0, 9n - 4)$$

since the sum of mutually independent Gaussian rvs is normally distributed with $\mathbb{E}\left[S_n\right] = 0$ and $\mathrm{Var}[S_n] = 1 + 9(n - 1) + 4 = 9n - 4$.

5.b. There a number of ways to solve this question:

A long way Fix θ in $\mathbb{R}$. For each $n = 1, 2, \ldots$,

$$\begin{aligned} \mathbb{E}\left[e^{i\theta S_n}\right] &= \mathbb{E}\left[e^{i\theta(\xi_1+3(\xi_2+\ldots+\xi_n)+2\xi_{n+1})}\right] \\ &= \mathbb{E}\left[e^{i\theta\xi_1}\right] \cdot \mathbb{E}\left[e^{i3\theta(\xi_2+\ldots+\xi_n)}\right] \cdot \mathbb{E}\left[e^{i2\theta\xi_{n+1}}\right] \end{aligned} \quad (1.21)$$

under the enforced independence assumptions so that

$$\mathbb{E}\left[e^{i\theta S_n}\right] = \Phi_\xi(\theta)\,\mathbb{E}\left[e^{i3\theta\Xi_{n-1}}\right]\Phi_\xi(2\theta). \quad (1.22)$$

However, the Central Limit Theorem (applied to the i.i.d. rvs $\{\xi_n,\ n = 1, 2, \ldots\}$) gives

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i3\theta\frac{\Xi_{n-1}}{\sqrt{n-1}}}\right] = e^{-\frac{(3\theta)^2}{2}}$$

while the continuity of characteristic functions at $\theta = 0$ implies

$$\lim_{n\to\infty} \Phi_\xi\left(\frac{\theta}{\sqrt{n-1}}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty} \Phi_\xi\left(2\frac{\theta}{\sqrt{n-1}}\right) = 1.$$

Together, upon combining these observations, we get

$$\lim_{n\to\infty} \mathbb{E}\left[e^{i\theta\frac{S_n}{\sqrt{n-1}}}\right] = e^{-\frac{9\theta^2}{2}}, \quad \theta \in \mathbb{R}$$

since

$$\mathbb{E}\left[e^{i\theta\frac{S_n}{\sqrt{n-1}}}\right] = \Phi_\xi\left(\frac{\theta}{\sqrt{n-1}}\right)\mathbb{E}\left[e^{i3\theta\frac{\Xi_{n-1}}{\sqrt{n-1}}}\right]\Phi_\xi\left(2\frac{\theta}{\sqrt{n-1}}\right)$$

for each $n = 1, 2, \ldots$ by virtue of (1.22). In other words,

$$\frac{S_n}{\sqrt{n-1}} \Longrightarrow_n 3U$$

where U is a standard Gaussian rv, and this is of course equivalent to

$$\frac{S_n}{\sqrt{n}} \Longrightarrow_n 3U.$$

Thus we can take $L = 3U$ and the sequence $b : \mathbb{N}_0 \to (0, \infty)$ to be $b_n = \sqrt{n}$.

A clever way For each $n = 1, 2, \ldots$, we note that

$$\begin{aligned} S_n &= \xi_1 + 3(\xi_2 + \ldots + \xi_n) + 2\xi_{n+1} \\ &=_{st} 3(\xi_1 + \ldots + \xi_n) \end{aligned} \quad (1.23)$$

(where $=_{st}$ denotes equal in distribution). Why? Because the rv $\xi_1+2\xi_{n+1}$ is independent of the rv $\xi_2 + \ldots + \xi_n$ and has the same distribution as 3ξ (which is also independent of the rv $\xi_2 + \ldots + \xi_n$). Since

$$\sqrt{n}\,(\xi_1 + \ldots + \xi_n) \Longrightarrow_n U$$

by the standard Central limit Theorem, whence

$$\sqrt{n}\left(3\left(\xi_1 + \ldots + \xi_n\right)\right) \Longrightarrow_n 3U$$

effortlessly!

6.

6.a. There are many ways to solve this problem

Using MMSE theory for Gaussian rvs Here the usual conditioning argument yields

$$\mathbb{E}\left[A^2B^2\right] = \mathbb{E}\left[A^2\mathbb{E}\left[B^2|A\right]\right]$$

with

$$\mathbb{E}\left[B^2|A\right] = \left(\mathbb{E}\left[B^2|A\right] - \mathbb{E}\left[B|A\right]^2\right) + \mathbb{E}\left[B|A\right]^2.$$

As a result,

$$\mathbb{E}\left[A^2B^2\right] = \mathbb{E}\left[A^2\left(\mathbb{E}\left[B^2|A\right] - \mathbb{E}\left[B|A\right]^2\right)\right] + \mathbb{E}\left[A^2\mathbb{E}\left[B|A\right]^2\right]. \tag{1.24}$$

We now use the fact that the rvs A and B are jointly Gaussian. Therefore, since the MMSE and LMMSE estimators coincide in this case, the Orthogonality Principle readily gives

$$\mathbb{E}\left[B|A\right] = \gamma A \quad \text{with} \quad \gamma = \frac{\rho}{\alpha^2}, \tag{1.25}$$

and the error rv $B - \mathbb{E}\left[B|A\right]$ is independent of the rv A. It is now plain[1] that

$$\begin{aligned}
\mathbb{E}\left[B^2|A\right] - \left(\mathbb{E}\left[B|A\right]\right)^2 &= \mathbb{E}\left[\left(B - \mathbb{E}\left[B|A\right]\right)^2|A\right] \\
&= \mathbb{E}\left[\left(B - \mathbb{E}\left[B|A\right]\right)^2\right] \\
&= \mathbb{E}\left[\left(B - \gamma A\right)^2\right] \\
&= \beta^2 - 2\gamma\rho + \gamma^2\alpha^2 \\
&= \beta^2 - \frac{\rho^2}{\alpha^2} \qquad (1.26)
\end{aligned}$$

so that

$$\mathbb{E}\left[A^2\left(\mathbb{E}\left[B^2|A\right] - \mathbb{E}\left[B|A\right]^2\right)\right] = \left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)\mathbb{E}\left[A^2\right]. \tag{1.27}$$

Next,

$$\mathbb{E}\left[A^2\mathbb{E}\left[B|A\right]^2\right] = \mathbb{E}\left[A^2\left(\gamma A\right)^2\right] = \gamma^2\mathbb{E}\left[A^4\right] \tag{1.28}$$

---

[1] Recall that $\mathbb{E}\left[B|A\right]$ is simply the mean of the conditional distribution of B given A.

with $\mathbb{E}\left[A^4\right] = 3\alpha^4$. Collecting we conclude from (1.24) that

$$\begin{aligned}
\mathbb{E}\left[A^2B^2\right] &= \mathbb{E}\left[A^2\left(\mathbb{E}\left[B^2|A\right] - \mathbb{E}\left[B|A\right]^2\right)\right] + \mathbb{E}\left[A^2\mathbb{E}\left[B|A\right]^2\right] \\
&= \left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)\mathbb{E}\left[A^2\right] + \gamma^2\mathbb{E}\left[A^4\right] \\
&= \left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)\alpha^2 + 3\gamma^2\alpha^4 \\
&= \left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)\alpha^2 + 3\rho^2 \\
&= \beta^2\alpha^2 + 2\rho^2. \qquad (1.29)
\end{aligned}$$

Using a representation for bivariate Gaussian rvs Recall that the pair $(A, B)'$ can be represented as

$$\begin{aligned}
A &= \alpha U \\
B &= \frac{\rho}{\alpha}\cdot U + \sqrt{\beta^2 - \frac{\rho^2}{\alpha^2}}\cdot V
\end{aligned}$$

where U and V are i.i.d. standard Gaussian rvs. It is now a simple exercise to compute

$$\begin{aligned}
\mathbb{E}\left[A^2B^2\right] &= \mathbb{E}\left[\alpha^2U^2\left(\frac{\rho}{\alpha}\cdot U + \sqrt{\beta^2 - \frac{\rho^2}{\alpha^2}}\cdot V\right)^2\right] \\
&= \alpha^2\mathbb{E}\left[U^2\left(\frac{\rho^2}{\alpha^2}U^2 + 2\frac{\rho}{\alpha}\sqrt{\beta^2 - \frac{\rho^2}{\alpha^2}}\cdot UV + \left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)V^2\right)\right] \\
&= \rho^2\mathbb{E}\left[U^4\right] + \alpha^2\left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)\mathbb{E}\left[U^2V^2\right] \qquad (1.30)
\end{aligned}$$

as w euse the fact that $\mathbb{E}\left[U^3V\right] = \mathbb{E}\left[U^3\right]\mathbb{E}\left[V\right] = 0$ by independence. since $\mathbb{E}\left[U^4\right] = 3$ and $\mathbb{E}\left[U^2V^2\right] = \mathbb{E}\left[U^2\right]\mathbb{E}\left[V^2\right] = 1$, it follows that

$$\mathbb{E}\left[A^2B^2\right] = 3\rho^2 + \alpha^2\left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)$$

and we get the desired answer!

A pedestrian way Another way to approach the problem was to recall that

$$\mathbb{E}\left[e^{i(aA+bB)}\right] = e^{-\frac{1}{2}(a,b)\boldsymbol{R}(a,b)'}, \quad a, b \in \mathbb{R}$$

and the answer can now be obtained by noting that

$$\mathbb{E}\left[A^2B^2\right] = \frac{\partial^2}{\partial a^2}\frac{\partial^2}{\partial b^2}\mathbb{E}\left[e^{i(aA+bB)}\right]\Big|_{a=b=0}.$$

6.b. For each $n = 0, 1, \ldots$, we have

$$\mathbb{E}[Y_n] = \mathbb{E}\left[(X_n)^2\right] = c(0)$$

and

$$\mathbb{E}\left[(Y_n)^2\right] = \mathbb{E}\left[(X_n)^4\right] = 3c(0)^2.$$

Next for each $n = 0, 1, \ldots$ and $k = 1, 2, \ldots$, we find

$$\begin{aligned} \mathbb{E}[Y_n Y_{n+k}] &= \mathbb{E}\left[(X_n)^2 (X_{n+k})^2\right] \\ &= c(0)^2 + 2c(k)^2 \end{aligned} \tag{1.31}$$

upon using Part **6.a** with $A = X_n$ and $B = X_{n+k}$ (so that $\alpha^2 = \beta^2 = c(0)$ and $\rho = c(k)$).

The sequence $\{Y_n,\ n = 0, 1, \ldots\}$ is wide-sense stationary. It is also strictly stationary because the Gaussian sequence $\{X_n,\ n = 0, 1, \ldots\}$, being wide-sense stationary, it is necessarily strictly stationary, hence the sequence $\{\varphi(X_n),\ n = 0, 1, \ldots\}$ is also strictly statioanry for any mapping $\varphi : \mathbb{R} \to \mathbb{R}$ (and in particular, $\varphi(x) = x^2$).
