ENEE 620
RANDOM PROCESSES
IN COMMUNICATION AND CONTROL
FALL 2016

SET THEORY:

Basic notation

The set of integers and the set of non-negative integers are denoted by $\mathbb{Z}$ and $\mathbb{N}$, respectively. So

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

and

$$\mathbb{N} = \{0, 1, \ldots\} = \{z \in \mathbb{Z} : \ z \geq 0\}.$$

It is sometimes convenient to write $\mathbb{N}_0$ to denote the set of all positive integers, i.e.,

$$\mathbb{N}_0 = \{1, 2, \ldots\} = \{n \in \mathbb{N} : \ n > 0\}.$$

Also, we use $\mathbb{R}$ to denote the collection of *all* real numbers – Think of $\mathbb{R}$ as the real line. We shall write $\mathbb{R}_+$ to denote the set of all non-negative real numbers, i.e.,

$$\mathbb{R}_+ = \{x \in \mathbb{R} : \ x \geq 0\}.$$

Countable vs. non-countable

A set S is said to be *countable* if there exists an *injective* mapping $T : S \to \mathbb{N}$ – To be injective means that if $T(x) = T(y)$ for x and y in S, then $x = y$ necessarily. Put differently, it is not possible for $x \neq y$ in S to satisfy $T(x) = T(y)$. In some literature, an injective mapping is also known as a *one-to-one* mapping. A set said that is *not* countable is said to be *uncountable*!

If S is countable, then the cardinality $|S|$ of S is either finite or infinite. If $|S|$ is finite, say $|S| = n$ for some non-negative integer n, we say that S is a finite set and we can represent it as $\{x_i, \ i = 1, \ldots, n\}$ by labeling its elements. If $|S| = \infty$, then S is said to be *countably infinite*, and we now represent it as $\{x_i, \ i \in I\}$ by indexing the elements of S through the index set I (which per force has to be countably infinite as well). Usually, but not always, I is taken to be $\mathbb{N}$ or $\mathbb{N}_0$.

It is easy to check the following: The sets $\{1, \ldots, m\}$ with $m = 1, 2, \ldots$, the set $\mathbb{Z}$ of all integers and the set $\mathbb{Q}$ of all rationals are countable sets, the last two being countably infinite. The unit interval $[0,1]$, the real line $\mathbb{R}$ and the plane $\mathbb{R}^2$ are not countable.

De Morgan's laws

Let E be an arbitrary set. If $\{A_i,\ i \in I\}$ is a collection of subsets of E, i.e., $A_i \subseteq E$ for each i in I, then

$$(\cup_{i\in I} A_i)^c = \cap_{i\in I} A_i^c$$

and

$$(\cap_{i\in I} A_i)^c = \cup_{i\in I} A_i^c$$

These two facts together are known as De Morgan's laws.

Distributivity

Let E be an arbitrary set. If $\{A_i,\ i \in I\}$ is a collection of subsets of E, then for any subset B of E, we have

$$B \cap (\cup_{i\in I} A_i) = \cup_{i\in I} (B \cap A_i)$$

and

$$B \cup (\cap_{i\in I} A_i) = \cap_{i\in I} (B \cup A_i) .$$

Inverting mappings

Consider a mapping $a : E \to F$ where E and F are arbitrary sets. We refer to E and F as the domain and range of a, respectively.

For each y in F, define

$$a^{-1}(y) = \{x \in E :\ a(x) = y\}.$$

The set $a^{-1}(y)$ is a subset of E and is often called the *pre-image* of y; it is the set of all elements in E that map to y. Of course it is possible to have $a^{-1}(y) = \emptyset$.

More generally, we define

$$a^{-1}(A) = \{x \in E :\ a(x) \in A\}, \quad A \subseteq F.$$

Note that

$$a^{-1}(\emptyset) = \emptyset.$$

Properties of inverting mappings

Consider a mapping $a : E \to F$ where E and F are arbitrary sets. If $\{A_i,\ i \in I\}$ is a collection of subsets of F, then the following facts hold:

(i) We have

$$a^{-1}\left(\cup_{i\in I}A_i\right) = \cup_{i\in I}a^{-1}\left(A_i\right)$$

and

$$a^{-1}\left(\cap_{i\in I}A_i\right) = \cap_{i\in I}a^{-1}\left(A_i\right).$$

(ii) If the sets $\{A_i,\ i \in I\}$ are *pairwise disjoint*, then the sets $\{a^{-1}\left(A_i\right),\ i \in I\}$ are also pairwise disjoint: Indeed, pick distinct i and j in I. By assumption, the sets are pairwise disjoint, i.e.,

$$A_i \cap A_j = \emptyset.$$

As a result,

$$a^{-1}\left(A_i\right) \cap a^{-1}\left(A_j\right) = a^{-1}\left(A_i \cap A_j\right) = a^{-1}\left(\emptyset\right) = \emptyset, \tag{1}$$

and the sets $a^{-1}\left(A_i\right)$ and $a^{-1}\left(A_j\right)$ are therefore disjoint.

(iii) Mapping inversion and complementarity commute as we have

$$a^{-1}\left(A^c\right) = \left(a^{-1}(A)\right)^c, \quad A \subseteq F.$$

This is a simple consequence of (ii) as we note the following: Since $A \cap A^c = \emptyset$, we have $a^{-1}\left(A\right) \cap a^{-1}\left(A^c\right) = \emptyset$. It then follows that $a^{-1}\left(A^c\right) \subseteq \left(a^{-1}\left(A\right)\right)^c$