Please work out the ten problems stated below – Show work and explain reasoning.

All rvs are defined on the same probability triple \((\Omega, \mathcal{F}, P)\).

1. __________________________________________________________________________

Compute the first two moments \(E[X]\) and \(E[X^2]\) (and the variance \(\text{Var}[X]\)) when the discrete rv \(X : \Omega \rightarrow \mathbb{R}\) is

a. a Binomial rv Bin\((n; p)\) with \(n = 1, 2, \ldots\) and \(0 < p < 1\),

b. a Poisson rv Poi\((\lambda)\) with \(\lambda > 0\),

c. a Geometric rv Geo\((p)\) with \(0 < p < 1\),

In each case explain why the expectations always exist.

2. __________________________________________________________________________

Using the change of variable formula, compute the expectation

\[
E\left[\frac{1}{1+Y^+}\right]
\]

when the rv \(Y : \Omega \rightarrow \mathbb{R}\) is

a. a Binomial rv Bin\((n; p)\) with \(n = 1, 2, \ldots\) and \(0 < p < 1\),

b. a Poisson rv Poi\((\lambda)\) with \(\lambda > 0\),

c. a Geometric rv Geo\((p)\) with \(0 < p < 1\),

In each case explain why the expectation \(E\left[\frac{1}{1+Y^+}\right]\) always exists.

3. __________________________________________________________________________

Show that

a. Claim (i) and Claim (ii) of the Monotone Convergence Theorem are equivalent.

b. Claim (i) and Claim (ii) of Fatou’s Lemma are equivalent.

4. __________________________________________________________________________
5. Let $N$ be a discrete rv with support $S_N$ contained in $\mathbb{N}_0$ (i.e., $\mathbb{P}[N \in \mathbb{N}_0] = 1$) and a finite second moment, i.e., $\mathbb{E}[N^2] < \infty$, and let $\{X_n, n = 1, 2, \ldots\}$ denote a collection of second-order rvs. Moreover, the rvs $\{X_n, n = 1, 2, \ldots\}$ have identical mean and variance, namely $\mu \equiv \mathbb{E}[X_1] = \mathbb{E}[X_2] = \ldots$ and $\sigma^2 \equiv \text{Var}[X_1] = \text{Var}[X_2] = \ldots$

Under the assumption that the rvs $\{N, X_n, n = 1, 2, \ldots\}$ are mutually independent, compute
\begin{itemize}
    \item[a.] the first moment $\mathbb{E}\left[\sum_{n=1}^{N} X_n\right]$.
    \item[b.] the variance $\text{Var}\left[\sum_{n=1}^{N} X_n\right]$.
\end{itemize}

6. We start with the rv $U : \Omega \to \mathbb{R}$ which has a symmetric distribution. Given are two Borel mappings $f, g : \mathbb{R} \to \mathbb{R}$ with the following properties: $f(-x) = -f(x)$ and $g(-x) = g(x)$ for all $x$ in $\mathbb{R}$. Assume that $\mathbb{E}[|f(U)|^2] < \infty$ and $\mathbb{E}[|g(U)|^2] < \infty$.

\begin{itemize}
    \item[a.] Show that $\text{Cov}[f(U), g(U)] = 0$, i.e., the rvs $X = f(U)$ and $Y = g(U)$ are always uncorrelated.
    \item[b.] Consider now the case when $f(x) = \sin(x)$ and $g(x) = \cos(x)$ for all $x$ in $\mathbb{R}$ (in which case $|f(x)|^2 + |g(x)|^2 = 1$). Show that $\text{Cov}[|f(U)|^2, |g(U)|^2] < 0$ with the implication that the rvs $|X|^2$ and $|Y|^2$ are not independent, and a fortiori $X$ and $Y$ cannot be independent!
\end{itemize}

7. With $0 < p < 1$, let $X(p), Y(p) : \Omega \to \mathbb{R}$ be a pair of independent Bernoulli rvs with $\mathbb{P}[X(p) = 1] = 1 - \mathbb{P}[X(p) = 0] = p$ and $\mathbb{P}[Y(p) = 1] = 1 - \mathbb{P}[Y(p) = 0] = p$.

\begin{itemize}
    \item[a.] Compute the covariance $\text{Cov}[|X(p) - Y(p)|, X(p) + Y(p)]$ between the rvs $|X(p) - Y(p)|$ and $X(p) + Y(p)$ as a function of $p$.
    \item[b.] Find all the values of $p$ in $(0, 1)$ such that rvs $|X(p) - Y(p)|$ and $X(p) + Y(p)$ are uncorrelated rvs.
    \item[c.] Find all the values of $p$ in $(0, 1)$ such that rvs $|X(p) - Y(p)|$ and $X(p) + Y(p)$ are independent rvs. [HINT: For the values of $p$ in Part $b$, compute the joint probability $\mathbb{P}[|X(p) - Y(p)| = 0, X(p) + Y(p) = 0]$ and compare it to the product $\mathbb{P}[|X(p) - Y(p)| = 0] \mathbb{P}[X(p) + Y(p) = 0]$].
\end{itemize}

8. Consider a rv $X : \Omega \to \mathbb{R}$.

\begin{itemize}
    \item[a.] Show that there always exists a scalar $M$ in $\mathbb{R}$ such that
    \[ \mathbb{P}[X \leq M] \geq \frac{1}{2} \quad \text{and} \quad \mathbb{P}[X \geq M] \geq \frac{1}{2}. \]
    Such a scalar is called a median for the probability distribution function of the rv $X$. Is such a scalar always unique? Explain!
    \item[b.] Let $F_X : \mathbb{R} \to [0, 1]$ denotes the probability distribution function of the rv $X$. If $F_X : \mathbb{R} \to [0, 1]$ is a strictly increasing and continuous function, show that there is only one median $t$ and it is characterized by $F_X(t) = \frac{1}{2}$. 
\end{itemize}
c. If $\mathbb{E} [|X|] < \infty$, then show that
\[ \mathbb{E} [|X - M|] \leq \mathbb{E} [|X - a|], \quad a \in \mathbb{R} \]
for every median $M$ of $X$.

9. Let $X : \Omega \rightarrow \mathbb{R}$ be a rv defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, with probability distribution function $F : \mathbb{R} \rightarrow [0, 1]$. With $a > 0$, define the functions $G_a : \mathbb{R} \rightarrow \mathbb{R}_+$ and $H_a : \mathbb{R} \rightarrow \mathbb{R}_+$ given by
\[ G_a(x) \equiv \frac{1}{a} \int_x^{x+a} F(t) \, dt, \quad x \in \mathbb{R} \]
and
\[ G_a(x) \equiv \frac{1}{2a} \int_{x-a}^{x+a} F(t) \, dt, \quad x \in \mathbb{R}. \]

a. Show that the functions $G_a : \mathbb{R} \rightarrow \mathbb{R}_+$ and $H_a : \mathbb{R} \rightarrow \mathbb{R}_+$ are both probability distribution functions.

b. Assume that the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to also carry a rv $U_a : \Omega \rightarrow \mathbb{R}$ which is uniformly distributed over the unit interval $[0, a]$ and which is independent of the rv $X$. Give an explicit construction for a rv, say $X_a$, which is defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ in terms of the rvs $X$ and $U_a$, and whose probability distribution function is $G_a$.

c. Can you still carry out a similar similar construction for $H_a$, i.e., give an explicit construction for a rv, say $X_a$, which is defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ in terms of the rvs $X$ and $U_a$, and whose probability distribution function is $H_a$? If not, what additional randomness do you need? Explain!

d. Assume that the rv $X$ has all its (integer) moments finite, namely $\mathbb{E} [|X|^r] < \infty$ for all $r = 1, 2, \ldots$. Compute the (integer) moments of the probability distribution functions $G_a$ and $H_a$ in terms of those of underlying probability distribution function $F : \mathbb{R} \rightarrow [0, 1]$ [HINT: Make use of your answer in Part b and Part c].

10. Let $X$ and $Y$ be two independent and identically distributed rvs defined on the same probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ with
\[ \mathbb{P} [X \leq x] = \mathbb{P} [Y \leq x] = 1 - e^{-\lambda x^+}, \quad x \in \mathbb{R} \]
for some $\lambda > 0$.

a. Compute
\[ \mathbb{P} [X \leq x, X + Y \leq s], \quad x, s \in \mathbb{R} \]
[HINT: What is the probability density function of the $\mathbb{R}^2$-valued rv $(X, Y)$?]

b. Determine whether the $\mathbb{R}^2$-valued rv $(X, X + Y)$ is of continuous type and in the affirmative, find its probability density function $f_{X, X+Y} : \mathbb{R}^2 \rightarrow \mathbb{R}_+$.

c. Are the rvs $X$ and $X + Y$ independent?