Please work out the ten (10) problems stated below – Show work and explain reasoning.

All rvs are defined on the same probability triple $(\Omega, \mathcal{F}, \mathbb{P})$.

1. For any rv $X : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{P}[X \in \mathbb{N}] = 1$, i.e., $X \in \mathbb{N}$ a.s., show that the inequality $\mathbb{P}[X > 0] \leq \mathbb{E}[X]$ always holds. This simple observation is the basis for the method of first moment often used in the theory of random graphs and in Combinatorics.

2. For any second-order rv $X : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{P}[X \geq 0] = 1$, i.e., $X \geq 0$ a.s., show that

$$\frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]} \leq \mathbb{P}[X > 0]$$

provided $\mathbb{E}[X^2] > 0$ [HINT: Note that $X = X \cdot 1[X > 0]$ a.s. and apply the Cauchy-Schwartz inequality]. This inequality is the starting point for the method of second moment often used in the theory of random graphs and in Combinatorics where it is applied to integer-valued count rvs in the form

$$\mathbb{P}[X = 0] \leq 1 - \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}.$$

3. With Hölder’s inequality generalizing the Cauchy-Schwartz inequality, Problem 2 suggests the following inequality:

   a. Consider a rv $X : \Omega \rightarrow \mathbb{R}$ such that $\mathbb{P}[X \geq 0] = 1$, i.e., $X \geq 0$ a.s. with $\mathbb{E}||X|^p| < \infty$ for some $p > 1$. With $q$ the conjugate of $p$, show that

$$\left( \frac{\mathbb{E}[X]}{(\mathbb{E}[X^p])^{\frac{1}{p}}} \right)^q \leq \mathbb{P}[X > 0]$$
provided $\mathbb{E} [|X|^p] > 0$ \textbf{[HINT:} Note that $X = X \cdot 1[X > 0]$ a.s. and apply H"older’s inequality].

\textbf{b.} Apply the result of Part \textbf{a} when $X$ is an exponential rv with unit parameter and $p$ is an integer, and explore how the bounds improve as $p$ increases.

\textbf{4.} Consider a rv $X : \Omega \to \mathbb{R}$ defined on probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E} [|X|] < \infty$.
\begin{enumerate}[a.]
\item Compute $\mathbb{E} [X | D]$ when $D = \Omega$.
\item Compute the conditional expectation of $X$ given the $\mathcal{F}$-partition $\{\emptyset, \Omega\}$.
\item Compute $\mathbb{E} [X | \mathcal{T}]$ where $\mathcal{T}$ denotes the trivial $\sigma$-field on $\Omega$.
\item Compute $\mathbb{E} [X | \mathcal{F}]$.
\end{enumerate}

\textbf{5.} Consider the two rv $X, Y : \Omega \to \mathbb{R}$ defined on probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that (i) the rvs $X$ and $Y$ are independent rvs, and (ii) each of the rvs is a standard normal. Set
\begin{equation*}
U \equiv XY \quad \text{and} \quad V \equiv \begin{cases} 
\frac{X}{Y} & \text{if } Y \neq 0 \\
0 & \text{if } Y = 0
\end{cases}
\end{equation*}

Using mostly \textit{conditioning} arguments answer the following questions:
\begin{enumerate}[a.]
\item Find the joint probability distribution of the rv $(U, V)$. Does it admit a probability density?
\item Find the joint probability distribution of the rv $U$. Does it admit a probability density?
\item Find the joint probability distribution of the rv $V$. Does it admit a probability density?
\item Are the rvs $U$ and $V$ independent?
\end{enumerate}

\textbf{6.} This problem involves \textbf{mutually independent} rvs $\{N, Y, X_n, \, n = 1, 2, \ldots\}$ which are all defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that (i) for each $n = 1, 2, \ldots$, the rv $X_n$ is exponentially distributed with parameter $\lambda > 0$, (ii) the rv $Y$ is uniformly distributed over the interval $(0, 1)$, and (iii) the rv $N$ is a geometric rv with parameter $a$ ($0 < a < 1$), say
\[ \mathbb{P} [N = k] = (1 - a)a^k, \quad k = 0, 1, \ldots \]

Using conditioning argument, answer the following questions:
\begin{enumerate}[a.]
\item Compute the expectation
\[ E(k) := \mathbb{E} [e^{-Y(X_1+\ldots+X_k)}] \]
for each $k = 0, 1, \ldots$ (with the convention that $X_1 + \ldots + X_k = 0$ if $k = 0$).
\item Compute the expectation
\[ \mathbb{E} [e^{-Y(X_1+\ldots+X_N)}], \]
\end{enumerate}

Carefully explain your calculations!
7. The rvs $X_1, \ldots, X_n$ are jointly Gaussian, e.g., with $X = (X_1, \ldots, X_n)'$, namely $X \sim N(\mu, R)$ for some vector $\mu$ in $\mathbb{R}^n$ and $n \times n$ covariance matrix $R$. With $a$ and $b$ elements in $\mathbb{R}^n$, define the $\mathbb{R}$-valued rvs $A$ and $B$ by

$$A \equiv a'X = \sum_{k=1}^{n} a_k X_k \quad \text{and} \quad B \equiv b'X = \sum_{k=1}^{n} b_k X_k.$$

**a.** Compute the characteristic function of the $\mathbb{R}^2$-valued rv $(A, B)'$, namely

$$\varphi(s, t) = \mathbb{E}[e^{i(sA + tB)}], \quad s, t \in \mathbb{R}.$$

Carefully explain your calculations!

**b.** With the help of your answer in Part a derive a necessary and sufficient condition on the parameters $\mu$, $a$, $b$ and $R$ for the rvs $A$ and $B$ to be independent. Carefully explain your calculations!

**c.** What form does this condition take when the rvs $X_1, \ldots, X_n$ are i.i.d. Gaussian rvs, say $X \sim N(\mu, \sigma^2 I_n)$ with $\sigma^2 > 0$?

8. Construct an $\mathbb{R}^2$-valued rv $X = (X_1, X_2)$ such that the $\mathbb{R}$-valued rvs $X_1$ and $X_2$ are each Gaussian but the $\mathbb{R}^2$-valued rv $X$ is not (jointly) Gaussian.

9. A rv $X$ is said to be a **Rice** rv with parameters $\alpha$ (in $\mathbb{R}$) and $\sigma > 0$ if

$$X =_{st} \sqrt{(\alpha + Y)^2 + Z^2} \quad (1.1)$$

where $Y$ and $Z$ are independent zero-mean Gaussian rvs with variance $\sigma^2$.

**a.** Use **conditioning** arguments to compute the probability distribution function of $X$, i.e., $\mathbb{P}[X \leq x]$ for all $x \geq 0$.

**b.** Use part a to conclude that the Rice rv is of continuous type and find its probability density function $f_X : \mathbb{R} \to \mathbb{R}^+$. 

10. A rv $X$ is said to be a **Rayleigh** rv with parameter $\sigma > 0$ if

$$X =_{st} \sqrt{Y^2 + Z^2} \quad (1.2)$$

where $Y$ and $Z$ are independent zero-mean Gaussian rvs with variance $\sigma^2$.

Without using or computing the probability density function of the rv $X$, show by **direct** arguments that if $X$ is a Rayleigh distribution with parameter $\sigma$, then $X^2$ is exponentially distributed with parameter $(2\sigma^2)^{-1}$ [**HINT:** Compute $\mathbb{E}[e^{-\theta X^2}]$ for a Rayleigh rv $X$ for $\theta \geq 0$.]