Please work out the ten (10) problems stated below – Show work and explain reasoning.

All rvs are defined on a probability triple \((\Omega, \mathcal{F}, \mathbb{P})\).

1. Consider a rv \(X : \Omega \rightarrow \mathbb{R}^p\) which is **symmetric** (under \(\mathbb{P}\)) in the sense that the rvs \(X\) and \(-X\) have the same probability distribution function (under \(\mathbb{P}\)). This is often written \(X =_{st} -X\) (with the understanding that all the probabilistic calculations are carried out under \(\mathbb{P}\)). Now consider a Borel mapping \(g : \mathbb{R}^p \rightarrow \mathbb{R}^q\).

   a. If the mapping is **even symmetric** in the sense that \(g(-x) = g(x)\) for all \(x\) in \(\mathbb{R}^p\), show that the rvs \(g(-X)\) and \(g(X)\) have the same probability distribution.

   b. If the mapping is **odd symmetric** in the sense that \(g(-x) = -g(x)\) for all \(x\) in \(\mathbb{R}^p\), show that the rv \(g(X)\) is itself a symmetric rv.

   Note that these conclusions do not require knowledge of the probability distribution function of the rv \(X\)!

2. Consider a pair of rvs \(X, Y : \Omega \rightarrow \mathbb{R}\). Use **direct** arguments to show that the following mappings \(\Omega \rightarrow \mathbb{R}\) are rvs:

   a. \(V = \max (X, Y)\)
   
   b. \(W = \min (X, Y)\)
   
   c. \(Z = \cos(2\pi X)\)

3. Consider a rv \(X : \Omega \rightarrow \mathbb{R}\) with probability distribution function \(F_X : \mathbb{R} \rightarrow [0, 1]\).

   a. How do you compute \(\mathbb{P} [X = x]\) for arbitrary \(x\) in \(\mathbb{R}\) in terms of the probability distribution function \(F_X : \mathbb{R} \rightarrow [0, 1]\)?

   b. It is customary to refer to a point \(x\) in \(\mathbb{R}\) to be a **point of discontinuity** for the probability distribution function \(F_X\) if \(F_X(x) - F_X(x-) > 0\). Show that the set of discontinuity points \(\{x \in \mathbb{R} : F_X(x) - F_X(x-) > 0\}\) is **countable**.
4. Consider a mapping $F : \mathbb{R} \to \mathbb{R}_+$ which is monotone non-decreasing, i.e., $F(x) \leq F(y)$ whenever $x < y$ in $\mathbb{R}$. The generalized inverse associated with $F$ is the mapping $F^- : \mathbb{R}_+ \to [-\infty, +\infty]$ given by

$$F^-(u) \equiv \inf \{ x \in \mathbb{R} : u \leq F(x) \}, \quad u \geq 0$$

with $F^-(u) = +\infty$ if the set $\{ x \in \mathbb{R} : u \leq F(x) \}$ is empty.

For the remainder of the problem assume $F$ to be a probability distribution function $F : \mathbb{R} \to [0, 1]$.

a. What is the value of $F^-(u)$ when $F(x-) \leq u < F(x)$ for some $x$ in $\mathbb{R}$ (which is a point of discontinuity for $F$)?

b. Find the generalized inverse associated with

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & \text{if } 0 \leq x < 10 \\ 1 & \text{if } 10 \leq x \end{cases}$$

with $0 < p < 1$. Draw the graph of $F^- : \mathbb{R}_+ \to [-\infty, \infty]$. Compute $F^-(F(x))$ for all $x$ in $\mathbb{R}$.

c. Find the generalized inverse associated with

$$F(x) = 1 - e^{-\lambda x^+}, \quad x \in \mathbb{R}$$

with $\lambda > 0$ and $x^+ = \max(0, x)$ for all $x$ in $\mathbb{R}$. Compute $F^-(F(x))$ for all $x$ in $\mathbb{R}$.

5. Consider a rv $X : \Omega \to \mathbb{R}$:

a. Give necessary and sufficient conditions on the probability distribution function $F_X : \mathbb{R} \to [0, 1]$ for the rv $X$ to be a symmetric rv $X : \Omega \to \mathbb{R}$ as introduced in Exercise 3.

b. Specialize your answer to Part a when $X$ is a discrete rv with support $S \subseteq \mathbb{R}$ and pmf $p = (p(x), \ x \in S)$: Give necessary and sufficient conditions on the pmf for the rv $X$ and its support $S$ to be a symmetric rv $X : \Omega \to \mathbb{R}$ as introduced in Exercise 3.

6. Consider rvs $X_1 : \Omega \to \mathbb{R}^{p_1}, \ldots, X_k : \Omega \to \mathbb{R}^{p_k}$. Show that the concatenated rv $X = (X_1, \ldots, X_k)$ taking values in $\mathbb{R}^p$ with $p = p_1 + \ldots + p_k$ is a discrete rv with support $S \subseteq \mathbb{R}^p$ if and only for each $\ell = 1, \ldots, k$, the rv $X_\ell : \Omega \to \mathbb{R}^{p_\ell}$ is a discrete rv with support $S_\ell \subseteq \mathbb{R}^{p_\ell}$. Can you relate the support $S$ to the supports $S_1, \ldots, S_k$? Do your answer depend on whether the rvs $X_1, \ldots, X_k$ are mutually independent?

7. Consider a mapping $F : \mathbb{R} \to \mathbb{R}_+$ which is monotone non-decreasing, i.e., $F(x) \leq F(y)$ whenever $x < y$ in $\mathbb{R}$. The generalized inverse associated with $F$ is the mapping $F^- : \mathbb{R}_+ \to [-\infty, +\infty]$ given by

$$F^-(u) \equiv \inf \{ x \in \mathbb{R} : u \leq F(x) \}, \quad u \geq 0$$

with $F^-(u) = +\infty$ if the set $\{ x \in \mathbb{R} : u \leq F(x) \}$ is empty.

For the remainder of the problem assume $F$ to be a probability distribution function $F : \mathbb{R} \to [0, 1]$.

a. What is the value of $F^-(u)$ when $F(x-) \leq u < F(x)$ for some $x$ in $\mathbb{R}$ (which is a point of discontinuity for $F$)?

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$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & \text{if } 0 \leq x < 10 \\ 1 & \text{if } 10 \leq x \end{cases}$$

with $0 < p < 1$. Draw the graph of $F^- : \mathbb{R}_+ \to [-\infty, \infty]$. Compute $F^-(F(x))$ for all $x$ in $\mathbb{R}$.

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5. Consider a rv $X : \Omega \to \mathbb{R}$:

a. Give necessary and sufficient conditions on the probability distribution function $F_X : \mathbb{R} \to [0, 1]$ for the rv $X$ to be a symmetric rv $X : \Omega \to \mathbb{R}$ as introduced in Exercise 3.

b. Specialize your answer to Part a when $X$ is a discrete rv with support $S \subseteq \mathbb{R}$ and pmf $p = (p(x), \ x \in S)$: Give necessary and sufficient conditions on the pmf for the rv $X$ and its support $S$ to be a symmetric rv $X : \Omega \to \mathbb{R}$ as introduced in Exercise 3.

6. Consider rvs $X_1 : \Omega \to \mathbb{R}^{p_1}, \ldots, X_k : \Omega \to \mathbb{R}^{p_k}$. Show that the concatenated rv $X = (X_1, \ldots, X_k)$ taking values in $\mathbb{R}^p$ with $p = p_1 + \ldots + p_k$ is a discrete rv with support $S \subseteq \mathbb{R}^p$ if and only for each $\ell = 1, \ldots, k$, the rv $X_\ell : \Omega \to \mathbb{R}^{p_\ell}$ is a discrete rv with support $S_\ell \subseteq \mathbb{R}^{p_\ell}$. Can you relate the support $S$ to the supports $S_1, \ldots, S_k$? Do your answer depend on whether the rvs $X_1, \ldots, X_k$ are mutually independent?

7. Consider a mapping $F : \mathbb{R} \to \mathbb{R}_+$ which is monotone non-decreasing, i.e., $F(x) \leq F(y)$ whenever $x < y$ in $\mathbb{R}$. The generalized inverse associated with $F$ is the mapping $F^- : \mathbb{R}_+ \to [-\infty, +\infty]$ given by

$$F^-(u) \equiv \inf \{ x \in \mathbb{R} : u \leq F(x) \}, \quad u \geq 0$$

with $F^-(u) = +\infty$ if the set $\{ x \in \mathbb{R} : u \leq F(x) \}$ is empty.

For the remainder of the problem assume $F$ to be a probability distribution function $F : \mathbb{R} \to [0, 1]$.

a. What is the value of $F^-(u)$ when $F(x-) \leq u < F(x)$ for some $x$ in $\mathbb{R}$ (which is a point of discontinuity for $F$)?

b. Find the generalized inverse associated with

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - p & \text{if } 0 \leq x < 10 \\ 1 & \text{if } 10 \leq x \end{cases}$$

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c. Find the generalized inverse associated with

$$F(x) = 1 - e^{-\lambda x^+}, \quad x \in \mathbb{R}$$

with $\lambda > 0$ and $x^+ = \max(0, x)$ for all $x$ in $\mathbb{R}$. Compute $F^-(F(x))$ for all $x$ in $\mathbb{R}$.
a. Consider two rvs $X, Y : \Omega \to \mathbb{R}$ defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, and as usual let $F_{X,Y} : \mathbb{R}^2 \to [0,1]$ denote their joint probability distribution function (under $\mathbb{P}$). If $X = Y$ a.s. (under $\mathbb{P}$), show that

$$F_{X,Y}(x,y) = H(\min(x,y)), \quad x, y \in \mathbb{R}$$

for some mapping $H : \mathbb{R} \to [0,1]$. Identify this mapping!

Next you are told that the function $F : \mathbb{R}^2 \to [0,1]$ is of the form

$$F(x,y) = K(\min(x,y)), \quad x, y \in \mathbb{R}$$

for some mapping $K : \mathbb{R} \to [0,1]$.

b. What properties should the mapping $K : \mathbb{R} \to [0,1]$ exhibit in order for the function $F : \mathbb{R}^2 \to [0,1]$ to be the joint probability distribution of a pair of rvs $U$ and $V$ defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$?

c. If the function $F : \mathbb{R}^2 \to [0,1]$ is indeed the joint probability distribution of a pair of rvs $U$ and $V$ defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, discuss whether the rvs $U$ and $V$ can be independent under $\mathbb{P}$?

d. Under the conditions obtained in Part b, is it always the case that $U = V$ a.s.? Explain.

8. Consider rvs $X_1 : \Omega \to \mathbb{R}^{p_1}$, ..., $X_k : \Omega \to \mathbb{R}^{p_k}$ defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Assume the rvs $X_1, \ldots, X_k$ to be mutually independent rvs

a. With Borel mappings $g_1 : \mathbb{R}^{p_1} \to \mathbb{R}^{q_1}$, ..., $g_k : \mathbb{R}^{p_k} \to \mathbb{R}^{q_k}$, show that the rvs $Y_\ell = g_\ell(X_\ell)$, $\ell = 1, \ldots, k$

are mutually independent.

b. Next, partition $\{1, \ldots, k\}$ into $r$ non-overlapping subsets $I_1, \ldots, I_r$, and set $a_s = \sum_{i \in I_s} p_i$ for $s = 1, \ldots, r$.

For each $s = 1, \ldots, r$, define the concatenated rv $X^{(s)} \equiv (X_i, i \in I_s)$ defined on $\Omega$ and taking values in $\mathbb{R}^{a_s}$. Explain why the rvs $X^{(1)}, \ldots, X^{(r)}$ are mutually independent.

9. Consider two rvs $X, Y : \Omega \to \mathbb{R}$ defined on some probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Assume the rv $X$ (resp. $Y$) to be a discrete rv with support $S_X \subseteq \mathbb{Z}$ (resp. $S_Y \subseteq \mathbb{Z}$) and pmf $p_X$ (resp. $p_Y$). Define the $Z : \Omega \to \mathbb{R}$ to be $Z \equiv X + Y$.

a. Explain why the rv $Z$ is a discrete rv, and relate its support $S_Z$ to the supports $S_X$ and $S_Y$ of the rvs $X$ and $Y$.

b. If we now assume the rvs $X$ and $Y$ to be independent, find the pmf of the rv $Z$ in terms of the pmfs $p_X$ and $p_Y$. 